5 Action of a groups on a set.

◇ **5.1** G is a group, X is a set. *Action of the group G on the set X* is a homomorphism $\phi\colon G \to S(X)$. $\phi(g)(x)$ is usually denoted as $\phi_g(x)$ or $g \cdot x$ or $g(x)$. Prove that this definition is equivalent to the following: *Action of the group G on the set X* is a map $\phi\colon G \times X \to X$ (we shall also denote $\phi(g, x)$ by $g \cdot x$), such that

1^o $\forall a, b \in G \quad \forall x \in X \qquad a \cdot (b \cdot x) = (ab) \cdot x.$

2^o $\forall x \in X \qquad e \cdot x = x$

◇ **5.2** Let G act on X, $x \in X$. The set $G \cdot x = \{g \cdot x, \quad g \in G\}$ is called the *orbit* of x. (So the action is transitive $\Leftrightarrow$ X is the orbit; in this case the orbit is unique.) The group $G_x = \{g \in G, \quad g \cdot x = x\}$ is called the *stabilizer* of x. $\operatorname{Ker}\phi = \{g \in G, \quad g \cdot x = x \quad \forall x \in X\} = \bigcap_{x \in X} G_x$ is called the *kernel* of this action. The action is called *faithful* if $\operatorname{Ker}\phi = \{e\}$.

(1) Prove that G_x is indeed a subgroup of G.

(2) Prove that there is a natural bijection between the orbit $G \cdot x$ and the quotient set G/G_x. Deduce from this that if the stabilizer G_x is a subgroup of finite index, then $|G \cdot x| = (G : G_x)$.

(3) Prove that $G_{g \cdot x} = gG_xg^{-1}$. (Prove first that for any subgroup $H \subset G$, $\forall g \in G$ the set $gHg^{-1} = \{ghg^{-1}, \quad h \in H\}$ is a subgroup in G.) Deduce from this that the stabilizers G_x and $G_{g \cdot x}$ are isomorphic.

(4) The set of all orbits of an action of a group G on a set X is called *the quotient set* X/G (sometimes denoted by $G\backslash X$). Let G and X be finite. For each orbit $c \in X/G$ fix one element $x_c \in c$. Prove the *orbit formula*:

$$|X| = \sum_{c \in X/G} (G : G_{x_c}).$$

◇ **5.3** Find the orbits of the following actions of D_6 and verify the orbit formula and the formula $|G \cdot x| = (G : G_x)$ for each orbit.

(1) D_6 acts on 6 vertices of the hexagon.

(2) D_6 acts on 9 diagonals of the hexagon.

(3) D_6 acts on 36 pairs (A, d), where A is a vertex of the hexagon and d is its short diagonal.

(4) D_6 acts on unordered pairs of different vertices of the hexagon.

(5) D_6 acts on unordered triples of different vertices of the hexagon.

◇ **5.4** Two actions of a group G on sets X and Y are called equivalent if $\exists$ such a bijection $f\colon X \to Y$ that the diagram

$$\begin{array}{ccc} G \times X & \xrightarrow{\mathrm{Id}_G \times f} & G \times Y \\ \big\downarrow{\scriptstyle \phi_X} & & \big\downarrow{\scriptstyle \phi_Y} \\ X & \xrightarrow{\ f\ } & Y \end{array}$$

is commutative. (This simply means that $\forall g \in G \;\; f(g \cdot x) = g \cdot f(x)$.)

(1) Prove that the actions of D_5 on the sides and on the vertices and on the diagonals of the pentagon are equivalent.

(2) Consider tree actions of D_6: on the sides, on the vertices and on the short diagonals of the hexagon. Prove that two of these three actions are equivalent and the third is not equivalent to them.

◇ **5.5** Let H be a subgroup in G. Consider the action of H on G by the left multiplication: $h \cdot g = hg$ ($h \in H$ and $g \in G$). Prove that it is the action (in the sense of 5.1) and that the orbits are right cosets. (This explains why the notation $H \backslash G$ is preferable here.)

◇ **5.6** Prove that the *conjugation* action of G on G defined by $g \cdot x = gxg^{-1}$ ($g, x \in G$) is the action (in the sense of ◇5.1) and that the fixed elements of this action are the elements of the center $Z(G)$. Prove that the kernel of this action $\operatorname{Ker} \phi$ (see the definition in ◇5.2) is also $Z(G)$.

◇ **5.7** (1) Prove that $\forall g, x \in G \ \ \operatorname{ord}(x) = \operatorname{ord}(gxg^{-1})$. Find a counterexample for the inverse statement.

(2) Two permutations in S_n are conjugate $\Leftrightarrow$ they are of the same cycle type.

(3) Find the orbits of the conjugation action of A_4 on itself. Use it to find a normal subgroup in A_4.

(4) Find the orbits of the conjugation action of A_5 on itself. Use it to prove that A_5 is simple.

◇ **5.8** Prove that $G/Z(G)$ can not be a cyclic group.

◇ **5.9** (1) Let $|G| = p^n$ where p is prime. Prove that $Z(G) \neq \{e\}$. (Use the conjugation action of G on itself and use ◇5.6 and ◇5.2.4.)

(2) Prove that if $|G| = p^2$ then G is abelian.

(3) Find a non-abelian group of order p^3, p — prime. (Find it in $\mathrm{SL}(3, \mathbb{F}_p)$.)

◇ **5.10** (1) $|G| = 15 \Rightarrow G$ is abelian. (Use the conjugation action of G on itself and on the set of its subgroups.)

(2) Find a non-abelian group of order 21. (Find it in $\mathrm{GL}(2, \mathbb{F}_7)$.)

(3) What is the difference between 15 and 21? For which pairs of prime p, q exists a non-abelian group of order pq?

◇ **5.11** (1) Write down the list of all known to you finite groups of order less then 30.

(2) Prove that your list is complete with the exception for the orders 12, 16, 24.

(3) (**) Try to classify groups of order 12, 16, 24.

◇ **5.12** Let $\mathbb{K}$ be a field. The multiplicative group $\mathbb{K}^*$ acts on $\mathbb{K} \times \mathbb{K}$ by $\lambda \cdot (x; y) = (\lambda x; \lambda y)$. The quotient set for this action is called *projective line over* $\mathbb{K}$ and is denoted by $\mathbb{P}^1(\mathbb{K})$. The orbit of $(x; y)$ is denoted by $(x : y)$. Prove that the mapping $(x : y) \mapsto \frac{x}{y}$ is a bijection between $\mathbb{P}^1(\mathbb{K}) \setminus \{(1 : 0)\}$ and $\mathbb{K}$.

◇ **5.13** Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2, \mathbb{K})$. Prove that $A \cdot (x : y) = (ax + by : cx + dy)$ defines an action of $\mathrm{PSL}(2, \mathbb{K})$ on $\mathbb{P}^1(\mathbb{K})$. Prove that $\forall M_1, M_2, M_3, N_1, N_2, N_3 \in \mathbb{P}^1(\mathbb{K})$ there exists unique $g \in \mathrm{PSL}(2, \mathbb{K})$ such that $g \cdot M_i = N_i$, $i = 1, 2, 3$.

◇ **5.14** (1) Prove that $\mathrm{PSL}(2, \mathbb{F}_3) \cong S_4$.

(2) Find all the subgroups of $\mathrm{PSL}(2, \mathbb{F}_5)$ isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. Study the conjugation action of $\mathrm{PSL}(2, \mathbb{F}_5)$ on the set of such subgroups. Use this action to prove that $\mathrm{PSL}(2, \mathbb{F}_5) \cong A_5$.

◇ **5.15** Consider an action of a group H on a group N by automorphisms, i.e. $\varphi : H \to \operatorname{Aut} N$. Define an operation $*$ on on $G = N \times H$ by $(n_1; h_1) * (n_2; h_2) = (n_1 \varphi_{h_1}(n_2); h_1 h_2)$. Prove that G becomes a group under $*$ and the sets $\widetilde{N} = \{(n; e_H),\ n \in N\}$ and $\widetilde{H} = \{(e_N; h),\ h \in H\}$ are subgroups in G, $\widetilde{N} \cong N$, $\widetilde{H} \cong H$, N is normal in G and G is a semidirect product of N and H. Prove that the conjugation action of $\widetilde{H}$ on $\widetilde{N}$ is exactly φ.