4. Normal subgroups and quotient groups.

Def 4.1 Let $H \lhd G$, define operation on G/H by $aH \cdot bH = abH$. The set G/H with this operation is called *quotient group.*

$\diamond$ **4.1** a) Prove that definition 4.1 is correct, i.e. if $aH = a_1H$, $bH = b_1H$, then $aH \cdot bH = a_1H \cdot b_1H$.
b) Prove that G/H is indeed a group under the operation defined in 4.1.

$\diamond$ **4.2** a) Prove that $\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}_n$. ($n\mathbb{Z}$ is the subgroup of all integers divisible by n.)
b) Prove that $\mathbb{Z}_{mn}$ has exactly one subgroup isomorphic to $\mathbb{Z}_m$: the subgroup $n\mathbb{Z}_{mn} = \{\bar{n}x, \quad x \in \mathbb{Z}_{mn}\}$. Prove that $\mathbb{Z}_{mn}/n\mathbb{Z}_{mn} \cong \mathbb{Z}_n$.

$\diamond$ **4.3** Let G and H be two groups. Prove that $\{e_G\} \times H$ is normal subgroup in $G \times H$ and $G \times H/(\{e_G\} \times H) \cong G$.

Def 4.2 Let $a = \begin{pmatrix} 1 & 2 & \dots & n \\ i_1 & i_2 & \dots & i_n \end{pmatrix} \in S_n$. An *inversion* is a pair of $k, l \in \{1, 2, \dots, n\}$ such that $k < l$ but $i_k > i_l$. The permutation a is called *even* (*odd*) if it has even (odd) number of inversions.

$\diamond$ **4.4** Define the mapping $\sigma : S_n \to \{\pm 1\}$ by the formula $\sigma(a) = (-1)^{(\text{number of inversions of } a)}$.
Prove that σ is a homomorphism. Useful statements for the proof:
a) Prove that any permutation is a product of transpositions.
b) Let $a, t \in S_n$ and t is a transposition. Prove that if a is even then at and ta are odd and if a is odd then at and ta are even.
c) Prove that if $a = t_1 t_2 \dots t_r$ where all the t_i are transpositions then $\sigma(a) = (-1)^r$.

Def 4.3 The set A_n of all even permutations ($= \operatorname{Ker} \sigma$ from 4.4) is called the *alternating* group.

$\diamond$ **4.5** Find $G/Z(G)$ (see problem 2.24) for
(a) $G = Q_8$, (b) $G = D_4$, (c) $G = A_4$, (d) $G = D_n$, (e) $G = A_5$.

$\diamond$ **4.6** Prove that $\operatorname{Int} G \cong G/Z(G)$.

$\diamond$ **4.7** Prove that $G/Z(G)$ can not be a cyclic group.

$\diamond$ **4.8** Find all normal subgroups and the corresponding quotient groups of
(a) Q_8, (b) D_4, (c) A_4, (b) D_n.

$\diamond$ **4.9** Let $H \lhd G$, $H' \lhd G'$, $H \cong H'$ and $G/H \cong G'/H'$. Does this imply that $G \cong G'$?

$\diamond$ **4.10** Let $H \lhd G$, define a mapping $\varphi_H : G \to G/H$ by $\varphi_H(g) = gH$. Prove that φ is a surjective homomorphism; it is called the *canonical homomorphism.* Prove that $\operatorname{Ker} \varphi_H = H$.

$\diamond$ **4.11** Let $f : G \to L$ be a homomorphism.
a) Prove that $G/\operatorname{Ker} f \cong \operatorname{Im} f$. (Note a useful corollary for finite groups: $|G| = |\operatorname{Ker} f| \cdot |\operatorname{Im} f|$)
b) Denote $\operatorname{Ker} f = H$. Prove that the mapping $\bar{f} : G/H \to L$ defined by $\bar{f}(gH) = f(g)$ is defined correctly. Prove that $\bar{f}$ is an injective homomorphism, $\operatorname{Im} f = \operatorname{Im} \bar{f}$ and the following diagram is commutative.

$$\begin{array}{ccc} G & \xrightarrow{f} & L \\ \downarrow \varphi_H & \nearrow_{\bar{f}} & \\ G/H & & \end{array}$$

$\diamond$ **4.12** a) Let $f : G \to L$ be a homomorphism, H is a subgroup in G and M is a subgroup in L. Prove that $f(H)$ is a subgroup in L and $f^{-1}(M)$ is a subgroup in G. ($f^{-1}(Y)$ is the set of all such $x \in G$ that $f(x) \in Y$.)
b) Prove that if M is normal then $f^{-1}(M)$ is normal.
c) Find an example when H is normal but $f(H)$ is not normal.
d) Prove that if f is surjective and H is normal then $f(H)$ is normal.
e) Let $K \lhd G$. Prove that there is one-to-one correspondence between normal subgroups of G containing K and normal subgroups of G/K.

$\diamond$ **4.13** Classify all groups of order 8.

$\diamond$ **4.14** Classify all groups of order $2p$, where p is prime.

$\diamond$ **4.15** a) Let K and H be two normal subgroups in G such that $G \supset K \supset H$. Prove that $H \lhd K$ and $G/K \cong (G/H)/(K/H)$.
b) Let K and H be two subgroups in G such that $G \supset K \supset H$ and K is normal in G and H is normal in K. Is it true that H is normal in G?

$\diamond$ **4.16** Let H and K be subgroups in G and $K \lhd G$.
a) Prove that $KH = \{kh, \quad k \in K, \ h \in H\} = \{hk, \quad k \in K, \ h \in H\}$ is a subgroup of G.
b) Prove that $K \cap H \lhd H$.
c) Prove that $KH/K \cong H/(K \cap H)$.
d) Let $K \cap H = \{e\}$. Is it true that $KH \cong K \times H$?

Def 4.4 Let H and K be subgroups in G, K normal in G, $K \cap H = \{e\}$ and $G = KH$ (this simply means that $\forall g \in G \ \ \exists h \in H, k \in K$ such that $g = kh$). Then we say that G is *semidirect product* of K and H.

$\diamond$ **4.17** Prove that under conditions of 4.4 $\ G/K \cong H$.

$\diamond$ **4.18** Prove that if G is semidirect product of K and H and both K and H are normal in G then $G \cong K \times H$.

$\diamond$ **4.19** Prove that D_n is semidirect product of C_n and $\{e, s\}$ where s is some reflection from D_n.

$\diamond$ **4.20** Prove that S_n is semidirect product of A_n and $\{e, t\}$ where s is a transposition.

$\diamond$ **4.21** Prove that S_4 is semidirect product of the four Klein group and S_3.

$\diamond$ **4.22** Let $\mathbb{E}(2)$ be the group of all isometries of the Euclidean plane, $\mathbb{E}_0(2)$ — the subgroup of all the isometries that preserve the orientation, $\mathbb{T}(2)$ — the subgroup of all translations. Let us fix some circle and denote by D_∞ the subgroup of all the isometries that preserve this circle.
a) Prove that $\mathbb{E}_0(2) \lhd \mathbb{E}(2)$ and $\mathbb{E}(2)$ is a semidirect product of $\mathbb{E}_0$ and $\{e, s\}$ where s is some reflection.
b) Prove that $\mathbb{T}(2) \lhd \mathbb{E}(2)$ and $\mathbb{E}(2)$ is a semidirect product of $\mathbb{T}(2)$ and D_∞ (and therefore $\mathbb{E}(2)/\mathbb{T}(2) \cong D_\infty$).
c) Prove that $\mathbb{E}_0(2)$ is a semidirect product of $\mathbb{T}(2)$ and S^1, where S^1 is the group of all rotations preserving some fixed circle (and therefore $\mathbb{E}_0(2)/\mathbb{T}(2) \cong S^1$).
d) State and prove the same results for $\mathbb{E}(3)$ — the group of all isometries of the Euclidean 3-space.

$\diamond$ **4.23** Give an example of a group G and its normal subgroup K such that G is not a semidirect product of K and H for any subgroup H of G.

$\diamond$ **4.24** Let $\mathbb{K}$ be some field. $\mathrm{GL}(n, \mathbb{K})$ is the group of all nondegenerate $n \times n$ matrices.
$\mathrm{SL}(n, \mathbb{K}) = \{A \in \mathrm{GL}(n, \mathbb{K}), \quad \det A = 1\}$, $\Lambda = \{\lambda E, \quad \lambda \in \mathbb{K}^*\}$ (E is the unit matrix).
a) Prove that $\mathrm{SL}(n, \mathbb{K}) \lhd \mathrm{GL}(n, \mathbb{K})$.
b) Prove that $\Lambda = Z(\mathrm{SL}(n, \mathbb{K})) = Z(\mathrm{GL}(n, \mathbb{K}))$; $\Lambda \cap \mathrm{SL}(n, \mathbb{K}) = Z(\mathrm{SL}(n, \mathbb{K}))$.

Def 4.5 Projective linear group is $\mathrm{PSL}(n, \mathbb{K}) = \mathrm{SL}(n, \mathbb{K})/(\Lambda \cap \mathrm{SL}(n, \mathbb{K}))$.

$\diamond$ **4.25** Prove that a) $\mathrm{PSL}(2, \mathbb{Z}_2) \cong S_3$; b) $\mathrm{PSL}(2, \mathbb{Z}_3) \cong A_4$; *c) $\mathrm{PSL}(2, \mathbb{Z}_5) \cong A_5$.