◇ **4.26** A subset $S \subset G$ is called the set of *generators* of the group G if $\forall g \in G$ can be represented as $g = s_1^{k_1} s_2^{k_2} \dots s_m^{k_m}$, where $s_i \in S$.

(1) Cyclic group = group with one generator.

(2) Find some minimal sets of generators for D_n and Q_8.

(3) Prove that S_n is generated by all the transpositions.

(4) Find two permutations generating S_n.

(5) Prove that A_n is generated by all the 3-cycles (i.e. cycles (ijk)).

(6) Prove that $GL(n, K)$ is generated by all matrices of the form $E + \lambda A(k,l)$, where $\lambda \in K$ and $A(k,l)_{i,j} = \delta_{k,i}\delta_{l,j}$, (for $k = l$, $\lambda \neq -1$).

◇ **4.27** Let A be some set that we shall call the alphabet set. Consider the set $W(A)$ of all the finite sequences of the form $a_1^{\varepsilon_1} a_2^{\varepsilon_2} \dots a_n^{\varepsilon_n}$, where $a_i \in A$, $\varepsilon_i = \pm 1$, $n \in \mathbb{N}$ (together with the empty sequence). Such sequences we shall call words. Reduction of a word $w = a_1^{\varepsilon_1} a_2^{\varepsilon_2} \dots a_n^{\varepsilon_n}$ is a new word, obtained from w by deleting two symbols $a_k^{\varepsilon_k} a_{k+1}^{\varepsilon_{k+1}}$, such that $a_k = a_{k+1}$ and $\varepsilon_k \varepsilon_{k+1} = -1$. Two words are called equivalent if they become equal after some number of reductions. The set of equivalence classes will be denoted by $F(A)$. The operation on $F(A)$ is defined in the most natural way: if $u = a_1^{\varepsilon_1} a_2^{\varepsilon_2} \dots a_n^{\varepsilon_n}$ and $v = b_1^{\mu_1} a_2^{\mu_2} \dots a_m^{\mu_m}$, then $u \cdot v = a_1^{\varepsilon_1} a_2^{\varepsilon_2} \dots a_n^{\varepsilon_n} b_1^{\mu_1} a_2^{\mu_2} \dots a_m^{\mu_m}$.

(1) Prove that $F(A)$ is a group. It is called *the free group, generated by the set A.*

(2) Let $S \subset G$ be some set of generators of the group G. Prove that there exists unique surjective homomorphism $\varphi_S \colon F(S) \to G$, such that $\varphi(s) = s$. (On the left side s is the word in $F(S)$ and on the right side it is the element of G.) $\operatorname{Ker} \varphi_S$ is called the *group of relations* for the generators S of G.

(3) Verify that if the set of generators of D_n is $S = \{a, s\}$, where a is the order n rotation and s is a reflection, then the group of relations $\operatorname{Ker} \varphi_S$ is generated by the three elements a^n, s^2, $sasa$ and their conjugates. (This simply means that if a group G is generated by such $\{a, s\}$ that $\operatorname{ord} a = n > 2$, $\operatorname{ord} s = 2$ and $\operatorname{ord} sa = 2$, then $G \cong D_n$.)

◇ **4.28** $a, b \in G$. The *commutator of the elements* a and b: $[a,b] = aba^{-1}b^{-1}$. The *commutator group* G' is the subgroup of G, generated by all the commutators of all the pairs of elements of G.

(1) $G' = \{[a_1, b_1] \cdot [a_2, b_2] \cdot \dots \cdot [a_m, b_m], \quad a_m, b_m \in G\}$

(2) Prove that $G' \lhd G$.

(3) Prove that G/G' is an abelian group.

(4) Prove that $\forall\ f : G \to F$ where f is a homomorphism and F is an abelian group $\operatorname{Ker} f \supset G'$. More precisely: $\forall$ homomorphism $f : G \to F$ with F abelian $\exists\ i : G/G' \to F$ such that the diagram

$$\begin{array}{ccc} G & \xrightarrow{f} & F \\ \downarrow^{\varphi} & \nearrow_{i} & \\ G/G' & & \end{array}$$

is commutative.

(5) Find Q_8'.

(6) Prove that $D_{2n+1}' = C_{2n+1}$, $D_{2n}' = C_n$.

(7) $(*)$ Prove that $S_n' = A_n$, $A_n' = A_n$. (Hint: calculate the commutator of two transpositions; two 3-cycles; represent a 3-cycle as a commutator of two even permutations; prove that A_n is generated by 3-cycles.)

(8) Let $S \subset G$ be some set of generators of an abelian group G. Prove that the group of relations $\operatorname{Ker} \varphi_S \supset F(S)'$.