HSE/Math in Moscow 2012-2013// Topology 2 // Problem sheet 6

Question 1. Recall that $\mathbb{R}P^n$ can be viewed as $S^n/\pm Id$. Let us identify $\{(x_0 : \cdots : x_{n-1} : 0) \in \mathbb{R}P^n\}$ with $\mathbb{R}P^{n-1}$ and $\{(x_0, \ldots x_{n-1}, 0) \in S^n\}$ with S^{n-1}.

(a) Prove that the homology map $H_n(S^n) \to H_n(S^n)$ induced by $-Id$ is multiplication by $(-1)^{n+1}$. [Hint: use question 9 from problem sheet 4.]

(b) Show that the quotient space $\mathbb{R}P^n/\mathbb{R}P^{n-1}$ is homeomorphic to S^n.

(c) Prove that the image of the homology map $H_n(S^n) \to H_n(\mathbb{R}P^n/\mathbb{R}P^{n-1})$ induced by $S_n \to S^n/S^{n-1} \to \mathbb{R}P^n/\mathbb{R}P^{n-1}$ has index 2 for odd n and is zero for even n.

(d) Using the map of couples $(S^n, S^{n-1} \to (\mathbb{R}P^n, \mathbb{R}P^{n-1})$ (or otherwise) show by induction on n that

$$H_*(\mathbb{R}P^n) = (\mathbb{Z}, \mathbb{Z}/2, 0, \mathbb{Z}/2, 0, \ldots, G)$$

where the last group G is in degree n and is $\mathbb{Z}$ for odd n odd and 0 for even n, and that for odd n the image of the homology map $H_n(S^n) \to H_n(\mathbb{R}P^n)$ induced by $S^n \to \mathbb{R}P^n$ has index 2.

Question 2. Recall that a *topological manifold of dimension* n is a Hausdorff space that has a countable dense set and such that every point has a neighbourhood homeomorphic to $\mathbb{R}^n$. Prove that no non-empty open subset of $\mathbb{R}^n$ is homeomorphic to an subset of $\mathbb{R}^m$ unless $m = n$. [Hint: try to compute the groups $H_*(U, U \setminus \{x\})$ where $U \subset \mathbb{R}^n$ is open and $x \in U$ using excision.] Deduce that the dimension of topological manifolds is well defined and that two topological manifolds can't be homeomorphic unless they are of the same dimension.

In the following question you may use the fact that the map $H_1(\mathbb{R}P^n) \to H_1(\mathbb{R}P^m)$ induced by any continuous map is zero if $n > m$. We will prove this later using the multiplication in cohomology. You may also use the fact that $H_1(\mathbb{R}P^n)$ is generated by the class represented by any non-contractible loop.

Question 3. (a) A map $f : S^n \to \mathbb{R}^m$ is called *even* if $f(-x) = f(x)$ for all $x \in S^n$ and *odd* $f(-x) = -f(x)$ for all $x \in S^n$. Prove that if $f : S^n \to \mathbb{R}^n$ is odd, there is an $x \in S^n$ such that $f(x) = 0$.

(b) Deduce from part (a) that at any given time there are at least two antipodal points on Earth where the temperature, air pressure and humidity are exactly the same.

(c) Show that no subset of $\mathbb{R}^n$ is homeomorphic to S^n. Deduce that there are no injective mape $\mathbb{R}^m \to \mathbb{R}^n$ if $m > n$.

Question 5. (a) Show that a continuous map $S^n \to S^n$ without fixed points is homotopic to the antipodal map $x \mapsto -x$.

(b) Deduce from part (a) and from part (a) of question 1 that if n is even then $\mathbb{Z}/2$ is the only non-trivial group that can act on S^n without fixed points.