HSE/Math in Moscow 2012-2013// Topology 2 // Problem sheet 7

Recall that a *CW-complex* is a topological space obtained inductively as follows. We start with X^0, which is a discrete space. Suppose we have constructed X^{n-1}. Then

$$X^n = X^{n-1} \cup_{\sqcup\varphi_\alpha} \sqcup D^n_\alpha$$

where D^n_α is an n-ball and $\varphi_\alpha : \partial D^n_\alpha \to X^{n-1}$ is a continuous map. In other words, X^n is obtained out of X^{n-1} by attaching some number of n-balls along continuous maps φ_α.

Equivalently, a CW-complex is a Hausdorff topological space X such that there are continuous maps $\gamma^n_\alpha : D^n \to X$ with the following properties:

- the restriction of γ^n_α to the interior of D^n_α is injective; the image of the interior of D^n_α under γ^n_α is called *an open cell of X.*
- X is a disjoint union of the of its open cells;
- (W) A subset of X is open if and only if its preimage under any γ^n_α is open in D^n_α, or, equivalently, if and only if its intersection with the closure of each cell is closed.
- (C) The closure of any cell is contained in the union of finitely many cells.

Note that conditions C and W are automatically satisfied if the number of the cells is finite. We do not prove that these two definitions are equivalent but you can look up a proof in Algebraic Topology (Appendix A) by A. Hatcher.

Question 1. Show that the following spaces are CW-complexes.
(a) Orientable compact surface of genus g.
(b) Nonorientable compact surface of genus g.
(c) Orientable compact surface of genus g with n boundary components.
(d) Nonorientable compact surface of genus g with n boundary components.
(e) S^n.
(f) $T^n = \mathbb{R}^n/\mathbb{Z}^n$.
(g) $\mathbb{R}P^n$.
(h) $\mathbb{C}P^n$.

Question 2. Recall that a topological space X is *normal* iff for any two closed $Y, Z \subset X$ such that $Y \cap Z = \varnothing$ there exist open U, V such that $Y \subset U, Z \subset V$ and $U \cap V = \varnothing$. Using the first definition of CW-complexes show that any CW-complex is normal and Hausdorff.

Question 3. Find a CW structure on $S^1 \vee S^2$ such that the closure of some cell is not a subcomplex.

Question 4. For each of the following spaces decomposed as unions of cells state which of the properties (C), (W) hold, if any.

(a) The unit disk in $\mathbb{R}^2$ represented as the union of its interior and the elements of the boundary considered as 0-cells.

(b) Let C_n be the circle in $\mathbb{R}^2$ with centre at $(\frac{1}{n}, 0)$ and of radius $\frac{1}{n}$. The union $\bigcup_n C_n$ is called the *Hawaiian earring.* It can be represented as the union of the origin $(0,0)$, which is the 0-cell, and the spaces $C_n \setminus \{(0,0)\}$, each of which is homeomorphic to an open interval.

Question 5. Show that for a CW-complex X the following statements are equivalent:
(a) X is connected.
(b) X is pathwise connected.
(c) X^1 is connected.

Question 6. Show that a CW-complex with one 0-cell and all the other cells having the same dimension is a wedge of spheres. (The topology on the wegde of infinitely many pointed spaces is introduced as follows: a subset of the wedge is open iff its intersection with each space is open.)