HSE/Math in Moscow 2012-2013// Topology 2 // Problem sheet 8

We have come to a point when we need to recall (or present) some of the basic properties of tensor products. We start with a commutative ring A.

The tensor product $M \otimes_A N$ of two A-modules M, N is defined as

$$FreeAb(M \times N) \Big/ \Big/ \left\langle \begin{array}{c} (m_1 + m_2, n) - (m_1, n) - (m_2, n), \\ (m, n_1 + n_2) - (m, n_1) - (m, n_2), \\ (am, n) - (m, an) \end{array} \right\rangle_{m, m_1, m_2 \in M, n, n_1, n_2 \in N, a \in A},$$

where $FreeAb(M \times N)$ is the free abelian group spanned by $M \times N$. The image of $(m, n) \in M \times N$ in $M \otimes_A N$ is denoted by $m \otimes_A n$. We can make $M \otimes_A N$ an A-module by setting $a \cdot (m \otimes_A n) = am \otimes_A n = m \otimes_A an$.

Question 1. Show that if M and N are generated as A-modules by $m_i, i \in I$, respectively $n_j, j \in J$, then $M \otimes_A N$ is generated by $m_i \otimes_A n_j, i \in I, j \in J$ as an A-module.

Question 2. a) Show that for all $m, n \in \mathbb{Z}_{>0}$ we have $\mathbb{Z}/n \otimes_{\mathbb{Z}} \mathbb{Z}/m \cong \mathbb{Z}/(m, n)$ where (m, n) denotes the greatest common divisor of m and n

b) Show that for all $n \in \mathbb{Z}_{>0}$ the tensor product $\mathbb{Z}/n \otimes_{\mathbb{Z}} M$ is zero provided M is an abelian group such that the map $M \to M$ defined as $m \mapsto nm, m \in M$is surjective.

Question 3. Show that if M, N, K are A-modules then the following A-modules are isomorphic

$$M \otimes_A (N \otimes_A K) \cong (M \otimes_A N) \otimes_A K,$$

$$M \otimes_A (N \oplus K) \cong (M \otimes_A N) \oplus (M \otimes_A K),$$

$$M \otimes_A N \cong N \otimes_A M,$$

$$A \otimes M \cong M.$$

[Hint: in each case try to construct an isomorphism by first defining it on the elements of the form $m \otimes_A n$; then check that the relations above are mapped to 0.]

Question 4. Show that for any A-modules M, N, K there is an isomorphism

$$\mathrm{Hom}_A(M \otimes_A N, K) \cong \mathrm{Hom}_A(M, \mathrm{Hom}_A(N, K)).$$

[Hint: take an $f : M \otimes_A N \to K$. Let us construct $F(f) : M \to \mathrm{Hom}_A(N, K)$ by setting $F(f)(m)(n) = f(m \otimes_A n)$; check that for all $m \in M$ the map $n \mapsto F(f)(m)(n)$ is a map of A-modules $N \to K$; check that if f is fixed this map depends on m A-linearly; so we get an arrow from the left hand side to the right hand side; check that it too is A-linear. The inverse mapping can be constructed in a similar way.]

Recall the recipe for computing $\mathrm{Tor}_{\mathbb{Z}}(M, N)$ where M, N are two abelian groups: we find a short exact sequence

$$0 \to P_1 \to P_0 \to M \to 0$$

with P_0, P_1 free. We then tensor the complex $[P_1 \to P_0]$ by N. The homology of the resulting complex is $M \otimes N$ in degree 0 and $\mathrm{Tor}_{\mathbb{Z}}(M, N)$ in degree 1. In this procedure M and N are interchangeable: we can resolve N first and tensor the result by M.

Question 5.

(a) Compute $\mathrm{Tor}_{\mathbb{Z}}(M, N)$ if M and N are cyclic abelian groups.

(b) Show that if M is any abelian group then $\mathrm{Tor}_{\mathbb{Z}}(M, \mathbb{Q}) = 0$.

Recall the recipe for computing $\mathrm{Ext}_{\mathbb{Z}}(M, N)$ where M, N are two abelian groups: we find a short exact sequence

$$0 \to P_1 \to P_0 \to M \to 0$$

with P_0, P_1 free. We then take the Hom from the complex $[P_1 \to P_0]$ to N. The homology of the resulting complex is $\mathrm{Hom}(M, N)$ in degree 0 and $\mathrm{Ext}_{\mathbb{Z}}(M, N)$ in degree 1. In this procedure M and N are *not* interchangeable.

Question 6.

(a) Show that $\mathrm{Ext}_{\mathbb{Z}}(M, N) = 0$ if $M = \mathbb{Z}$ or $N = \mathbb{Q}$.

(b) Compute $\mathrm{Ext}_{\mathbb{Z}}(M, N)$ if M and N are cyclic abelian groups.

Note that Question 6 allows one to compute the Ext of any two groups from the following list $\mathbb{Z}, \mathbb{Z}_n, \mathbb{Q}$, apart from $\mathrm{Ext}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z})$. This group is in fact enormous but we do not prove it here.