HSE/Math in Moscow 2012-2013// Topology 2 // Problem sheet 5

The first few problem sheets started with general topology problems. But now we'll say goodbye to general topology (at least for a while) and proceed to our main topic, the (integral) homology groups.

Singular homology groups

Question 1. Compute the singular homology groups of the complex projective space $\mathbb{C}P^n$ (see problem sheet 3 for the definition of $\mathbb{C}P^n$). [Hint: let us denote the equivalence class of $(z_0, \dots, z_n) \in \mathbb{C}^{n+1} \setminus \{0\}$ as $(z_0 : \dots : z_n)$; take $U_1 = \mathbb{C}P^n \setminus \{(0 : \dots : 0 : 1)\}, U_2 = \{(z_0 : \dots : z_{n-1} : 1)\}$; show that U_1 is homotopy equivalent to $\mathbb{C}P^{n-1}$ and that U_2 is homeomorphic to $\mathbb{C}^n$, and then apply the Mayer-Vietoris exact sequence.]

Question 2. In this question our task is to calculate the homology of a compact orientable surface, possibly with boundary.

a) Show that the homology of the 2-torus $T^2 = S^1 \times S^1$ is given by $H_i(T^2) \cong \mathbb{Z}$ if $i = 0, 2$, $\mathbb{Z}^2$ if $i = 1$ and 0 if $i \neq 0, 1, 2$. [Hint: one could e.g. represent one of the factors S^1 as a union of two intervals and then times this with the other S^1.]

b) We shall say that a class $\in H_1(X)$ is *represented by a loop* $s : S^1 \to X$ if it is equal $s_*(c(\gamma))$ where $c(\gamma)$ is the canonical generator of $H_1(S^1)$, see question 7 c) from the previous problem sheet. Describe loop representatives for the elements of some basis of $H_1(T^2)$.

c) Let X be the 2-torus with a small open disk removed and let S be the boundary of X. Using the results from part a) show that the map $H_1(S) \to H_1(X)$ induced by the inclusion $S \to X$ is zero. Deduce that $H_1(X) \cong H_1(T^2)$.

d) Let X be the connected sum of g 2-tori (see blackboard). Alternatively, X can be obtained as a result of identifying the edges of a regular $4g$-gon as shown on the blackboard. It is not too hard to prove that the resulting spaces are homeomorphic but we will not do this. Let Y be X with a small open disk removed and let S be the boundary of Y. Prove that Y is homotopy equivalent to a wedge of $2g$ circles and deduce that $H_1(Y) \cong \mathbb{Z}^{2g}$.

Show by induction on g that

i) $H^i(X) \cong \mathbb{Z}$ if $i = 0, 2$, $\mathbb{Z}^{2g}$ if $i = 1$ and 0 if $i \neq 0, 1, 2$;

ii) the map $H_1(S) \to H_1(Y)$ induced by the inclusion $S \to X$ is zero.

Describe loop representatives for the elements of some basis of $H_1(X)$.

e) Let X be the closed unit disk in $\mathbb{R}^2$ with n disjoint small open disks removed from the interior. Show that X is homotopy equivalent to a wedge of n circles and find loop representatives for the elements of some basis of $H_1(X)$. Note that X contains the unit circle S^1 (since the disks we remove do not intersect it). Compute the image of the canonical generator of $H_1(S^1)$ under the map induced by the inclusion $S^1 \subset X$.

f) Using the results of parts d) and e) calculate the homology groups of a connected sum of g tori with n disjoint open disks removed.

Question 3. Non-orientable surfaces can be tackled in a similar way:

a) Show that the real projective plane $\mathbb{R}P^2$ with a small open disk removed is homeomorphic to the Möbius strip.

b) Let X be the closed Möbius strip and let S be its boundary. Show that both $H_1(X)$ and $H_1(S)$ are isomorphic to $\mathbb{Z}$ and find loop representatives for generators of both groups. Show that using these generators the map $H_1(S) \to H_1(X)$ induced by the inclusion can be written as $x \mapsto \pm 2x$.

c) Calculate the homology groups of the real projective plane.

d) Let X be the connected sum of g real projective planes (see blackboard). Alternatively, X can be obtained as a result of identifying the edges of a regular $2g$-gon as shown on the blackboard. It is not too hard to prove that the resulting spaces are homeomorphic but we will not do this. Let Y be X with a small open disk removed and let S be the boundary of Y. Prove that Y is homotopy equivalent to a wedge of g circles and deduce that $H_1(Y) \cong \mathbb{Z}^g$.

Show by induction on g that

i) $H^i(X) \cong \mathbb{Z}$ if $i = 0$, $\mathbb{Z}^{g-1} \oplus \mathbb{Z}/2$ if $i = 1$ and 0 if $i \neq 0, 1$;

ii) the map $H_1(S) \to H_1(Y)$ induced by the inclusion $S \to X$ takes a generator of $H_1(S)$ to twice some generator of $H_1(Y)$.

Describe loop representatives for the elements of some basis of $H_1(X)$.

e) Using the results of part d) and part e) of the previous question calculate the homology groups of a connected sum of g real projective planes with n disjoint open disks removed.

Homological algebra

Question 4. Let $C^i_*, i = 1, 2, 3$ be complexes with differentials ∂_i. Suppose $f^1, g^1 : C^1_* \to C^2_*$ and $f^2, g^2 : C^2_* \to C^3_*$ are maps of complexes and $D_1 : C^1_* \to C^2_{*+1}$ and $D_2 : C^2_* \to C^3_{*+1}$ be homotopies between f^1 and g^1 and between f^2

and g^2 respectively, i.e.

$$f^1 - g^1 = \partial_2 D_1 + D_1 \partial_1, f^2 - g^2 = \partial_3 D_2 + D_2 \partial_2.$$

Construct a homotopy between $f^2 \circ f^1$ and $g^2 \circ g^1$, i.e. a map $D : C^1_* \to C^3_{*+1}$ such that

$$f^2 \circ f^1 - g^2 \circ g^1 = \partial_3 D + D \partial_1.$$

[Hint: start by writing

$$f^1 = g_1 + \partial_2 D_1 + D_1 \partial_1,$$
$$f^2 = g_2 + \partial_3 D_2 + D_2 \partial_2$$

and then compose.]