HSE/Math in Moscow 2012-2013// Topology 2 // Problem sheet 9

Recall that in the lectures we introduced the *cellular chain complex* $\mathcal{C}_*(X)$ of a CW-complex X. We have $\mathcal{C}_n(X) = H_n(X^n, X^{n-1})$ and the differential $\partial : \mathcal{C}_n(X) \to \mathcal{C}_{n-1}(X)$ is the connecting homomorphism of the triple (X^n, X^{n-1}, X^{n-2}). We have shown that the homology of $\mathcal{C}_*(X)$ is isomorphic to $H_*(X)$ functorially with respect to cellular maps.

Each group $\mathcal{C}_n(X) = H_n(X^n, X^{n-1})$ is isomorphic to $H_n(\vee_\alpha S^n_\alpha)$ where α runs through the set of n-cells of X. Choose a generator $e_n \in H_n(D^n, S^{n-1})$. Recall that we have maps $\gamma^n_\alpha : D^n \to X^n$ that take every boundary point to X^{n-1}. (These maps are in fact part of the CW-structure.) So for each α we have a generator $e^n_\alpha = (\gamma^n_\alpha)_*(e_n) \in H_n(X^n, X^{n-1})$. To calculate its image under the differential $\mathcal{C}_n(X) \to \mathcal{C}_{n-1}(X)$ we can do the following.

The map γ^n_α induces a map $S^{n-1} \to X^{n-1} \to X^{n-1}/X^{n-2} = \vee_\beta S^{n-1}_\beta$ where β runs through the $n-1$-cells of X; this map will be denoted as φ^n_α. Let f_{n-1} be the image of e_n under the connecting homomorphism of the pair (D^n, S^{n-1}). Then $\partial e^n_\alpha = (\varphi^n_\alpha)_*(f_{n-1})$.

Question 1. Choose f_1 to be the class of the loop $t \mapsto e^{2\pi i t}, t \in [0,1]$ and choose $e_2 \in H_2(D^2, S^1)$ accordingly. Using the CW-structure you have constructed in question 1, problem sheet 7, compute the resulting cellular chain complex for

(a) a orientable genus g surface without boundary.

(b) a non-orientable genus g surface without boundary.

[Hint: start with the 2-torus, respectively, the real projective plane.]

One can also define cellular chain complexes with coefficients in an abelian group M: for a CW-complex X we set $\mathcal{C}_n(X, M) = H_n(X^n, X^{n-1}, M)$ and the rest is completely similar to what happens in the case $M = \mathbb{Z}$. Note that since the homology of a sphere is a free abelian group we have $\mathcal{C}_n(X, M) = \mathcal{C}_n(X) \otimes M$. Just as in the case $M = \mathbb{Z}$ one shows that $H_*(\mathcal{C}_*(X, M)) \cong H_*(X, M)$ functorially with respect to cellular maps.

Similarly, one defines the *cellular cochain complex* of X by setting $\mathcal{C}_n(X, M) = H_n(X^n, X^{n-1}, M)$. The cohomology of the resulting complex will be isomorphic to $H^*(X, M)$ and we have

$$\mathcal{C}^*(X, M) \cong \mathrm{Hom}(\mathcal{C}_n(X), M)$$

as complexes.

Question 2. Using the results from problem sheet 5 calculate the homology with coefficients in $\mathbb{Z}/2$ of a non-orientable genus g surface

(a) using the universal coefficients formula

(b) using the cellular chain complex

and compare the results.

Question 3. Let X be S^n and set $Y = X \cup_f D^{n+1}$ where $f : \partial D^{n+1} = S^n \to X$ be a map that takes some generator to m times a generator of $H_n(X)$. Set $Z = Y/X$.

(a) Equip X, Y, Z with CW-structures such that the maps $i : X \to Y, p : Y \to Z$ are cellular maps.

(b) Calculate the corresponding cellular chein complexes (with the differentials) and the chain maps induced by i and p.

(c) Show that p induces a zero map of $H_{>0}(-, \mathbb{Z})$ but not of $H_{n+1}(-, \mathbb{Z}/m)$ nor $H_{n+1}(-, \mathbb{Z}/m)$. Deduce that the splittings in the universal coefficients formulae are not functorial.

Question 4. Using the Künneth formulae and Question 1 calculate the integral homology of $\mathbb{R}P^2 \times \mathbb{R}P^2$. Is it true that for all n we have

$$H_n(\mathbb{R}P^2 \times \mathbb{R}P^2, \mathbb{Z}) \cong \Sigma_{p+q=n} H_p(\mathbb{R}P^2, \mathbb{Z}) \otimes H_q(\mathbb{R}P^2, \mathbb{Z})?$$

Question 5. Using excision show that if M is a manifold and $x \in M$ then $H_n(M, M \setminus \{x\}) \cong \mathbb{Z}$.

Recall that a compact connnected m-manifold M is *orientable* if $H_m(M) \cong \mathbb{Z}$. An *orientation* of an orientable manifold is a choice of one of the two generators of this group. Once chosen this generator is denoted $[M]$ and called the *fundamental class of* M) such that for any $x \in M$ the image of $[M]$ under the map $H_n(M) \to H_n(M, M \setminus \{x\}) \cong \mathbb{Z}$ is a generator.

If N is a connected oriented n-submanifold of an oriented compact connected m-manifold M then the image of $[N]$ in $H_n(M)$ is called the *homology class* represented by N. There is the Poincaré isomorphism

$$H^i(M) \cong H_{m-i}(M).$$

If under this isomorphism cohomology classes c_1, c_2 go to the homology classes represented by oriented submanifolds N_1, N_2 then $c_1 \smile c_2$ goes to the class represented by $N_1 \cap N_2$, provided N_1, N_2 intersect transversally.

The same holds when we consider homology and cohomology with coefficients in any commutative ring A, except that when $A = \mathbb{Z}/2$ everything simplifies: for every compact m-manifold M we have $H_m(M, \mathbb{Z}/2) \cong \mathbb{Z}/2$ and clearly the latter group has just one generator.

Question 6. Calculate the cup product in
(a) $H^*(\mathbb{C}P^n, \mathbb{Z})$;
(b) $H^*(\mathbb{R}P^2, \mathbb{Z}/2)$.

Question 7. In the lectures we have computed the cup product in the homology of orientable surface. Try to do the same for the cohomology with coefficients mod 2 of non-orientable surfaces.