HSE/Math in Moscow 2012-2013// Topology 2 // Problem sheet 4

General topology continued

Question 1. Let X be a topological space. We introduce an equivalence relation on X by declaring $x \sim y, x, y \in X$ if and only if there is a path $\gamma : [0,1] \to X$ such that $\gamma(0) = x, \gamma(1) = y$.

a) Show that this is indeed an equivalence relation. The resulting equivalence classes are called the *path components* of X.

b) Show that the path components are path connected and that any path connected subspace of X is included in some path component.

Question 2. Let X be a topological space. Let us introduce another equivalence relation on X by declaring $x \sim y, x, y \in X$ if and only if there is a connected subspace of X that contains both x and y.

a) Show that this is indeed an equivalence relation. [Hint: prove that if Y_1, Y_2 are connected and $Y_1 \cap Y_2 \neq \varnothing$ then $Y_1 \cup Y_2$ is connected.] The resulting equivalence classes are called the *connected components* of X.

b) Show that the connected components are indeed connected and that any connected subspace of X is included in some connected component.

Since every subset of a topological space X is dense in its closure (prove it!), the closure of a connected subset of X is again connected by question 2 a) from problem sheet 2, and so the connected components of X are closed. They needn't be open:

Question 3. Take $X = \mathbb{Q}$ with the topology induced from $\mathbb{R}$. Show that the connected components are precisely the 1-element subsets of X and that they are not open.

Question 4. Take X to be the space from question 7 b) from problem sheet 1 (the closure of the graph of $x \mapsto \sin \frac{1}{x}$). What are the path components of X?

We know from problem list 1 that any path connected space is connected. So any path component is included in some connected component. But it may happen that the same connected component contains several path components. However, in most examples we'll be interested in this does not happen and connected components and path components are the same.

In particular, we say that a topological space X is *locally path connected* if any point of X has a path connected neighbourhood. (Recall that a *neighbourhood* of $x \in X$ is any open subset that contains x.)

Question 5. Show that the path components of a locally path connected space are open. Deduce that each connected component contains precisely one path component, and hence that connected components and path components coincide.

Representing homology classes by paths

Let X be a topological space. Two singular 1-chains $c_1, c_2 \in C_1(X)$ are *homologous* iff there is a $c \in C_2(X)$ such that $c_1 - c_2 = \partial c$. If this is the case we write $c_1 \sim c_2$.

Starting from a path $\gamma : [0,1] \to X$ we can construct a 1-chain $c(\gamma)$ using a natural identification of $\triangle^1 \cong [0,1]$ (namely, we identify $(1,0) \in \triangle^1$ with 0 and $(0,1)$ with 1).

Question 6. a) Let $\gamma_1, \gamma_2 : [0,1] \to X$ be paths such that $\gamma_1(1) = \gamma_2(0)$. Let $\gamma : [0,1] \to X$ be given by the following formula: $\gamma(t) = \gamma_1(2t)$ for $t \leq \frac{1}{2}$ and $\gamma(t) = \gamma_1(2t-1)$ for $t \geq \frac{1}{2}$. Show that γ is well defined and continuous; it is called the *concatenation* of γ_1 and γ_2. Also show that $c(\gamma) \sim c(\gamma_1) + c(\gamma_2)$.

b) We say that paths $\gamma_1, \gamma_2 : [0,1] \to X$ are *homotopic rel boundary* iff $\gamma_1(0) = \gamma_2(0), \gamma_1(1) = \gamma_2(1)$ and there is a homotopy $F : [0,1] \times [0,1] \to X$ such that $F(t,0) = \gamma_1(t), F(t,1) = \gamma_2(t), t \in [0,1]$ and $F(0,s) = \gamma_1(0) = \gamma_2(0), F(1,s) = \gamma_1(1) = \gamma_2(1)$ for all $s \in [0,1]$. In other words, the homotopy transforms one path into the other while keeping the endpoints fixed. Show that if paths $\gamma_1, \gamma_2 : [0,1] \to X$ are homotopic rel boundary, then $c(\gamma_1) \sim c(\gamma_2)$.

c) Let $\gamma : [0,1] \to X$ be a path and let $\bar{\gamma} : [0,1] \to X$ be defined as $\bar{\gamma}(t) = \gamma(1-t), t \in [0,1]$. Show that $c(\bar{\gamma}) + c(\gamma) \sim 0$.

Question 7. Now take $X = S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. In the lectures we have found that $H_1(S^1) \cong \mathbb{Z}$. Now we would like to visualise this result.

Let $\gamma_1 : [0,1] \to X$ and $\gamma_2 : [0,1] \to X$ be defined as follows:

$$\gamma_1(t) = e^{\pi i t}, \gamma_2(t) = e^{\pi i (t+1)}.$$

Set $U_1 = S^1 \setminus \{-i\}, U_2 = S^1 \setminus \{i\}$ and $\mathcal{U} = \{U_1, U_2\}$. Recall that in the lectures we have defined the complex $C_*^{\mathcal{U}}(X)$ as the subcomplex of $C_*(X)$ generated as an abelian subgroup by singular simplices whose image is included in one of open sets U_1, U_2.

a) Prove that $c(\gamma_1) + c(\gamma_2))$ is a cycle, i.e., $\partial(c(\gamma_1) + c(\gamma_2)) = 0$.

b) By considering the connecting homomorphism in the long exact sequence constructed from the short exact sequence of complexes

$$0 \to C_*(U_1 \cap U_2) \xrightarrow{+} C_*(U_1) \oplus C_*(U_2) \xrightarrow{-} C_*^{\mathcal{U}}(X) \to 0$$

prove that the homology class of $c(\gamma_1) + c(\gamma_2)$ is a generator of $H_1(S^1)$.

c) Deduce from the previous question that the homology class of $c(\gamma)$ also generates $H_1(S^1)$ where $\gamma : [0,1] \to X$ is given by

$$\gamma(t) = e^{2\pi i t}.$$

d) For $z \in X$ set $f(z) = \bar{z}, f_n(z) = z^n$ (here n is a positive integer). Compute the mappings $f_*, (f_n)_* : H_1(S^1) \to H_1(S^1)$ induced by f and f_n respectively.

The homology of the sphere

Recall that

$$S^n = \left\{ (x_1, \ldots, x_{n+1}) \mid \sum_{k=1}^{n+1} x_k^2 = 1 \right\}.$$

Question 8. Generalising the argument from the lectures show by induction on n that $H_i(S^n) \cong \mathbb{Z}$ if $i = 0, n$ and 0 otherwise. [There is one slightly delicate point here: how to deal with H_0? Fortunately, this is quite easy: if spaces X, Y are both path connected, the map $H_0(X) \to H_0(Y)$ induced by any continous map $f : X \to Y$ is an isomorphism: we have seen in the lectures that both groups are isomorphic to $\mathbb{Z}$ and are generated by the homology class of any singular 0-simplex. So, when both open subsets, as well as their intersection, are path connected, the Mayer-Vietoris sequence in fact terminates at H_1, meaning that the connecting homomorphism $H_1 \to H_0$ is zero.]

Question 9. Set $U_1 = S^n \setminus (0, \ldots, -1), U_2 = S^n \setminus (0, \ldots, 1)$ and let $f : S^n \to S^n$ be the reflection in the $x_1 = 0$ plane, i.e.,

$$f(x_1, x_2, \ldots, x_{n+1}) = (-x_1, x_2, \ldots, x_{n+1}).$$

Using the Mayer-Vietoris sequence show by induction on n that f induces minus identity in $H_n(S^n)$.