HSE/Math in Moscow 2012-2013// Topology 2 // Problem sheet 3

General topology continued

Let $\sim$ be an equivalence relation on a topological space X. We topologise the set of the equivalence classes $X/\sim$ by declaring $U \subset X/\sim$ open if and only if $p^{-1}(U) \subset X$ is open where $p : X \to X/\sim$ is the map that takes an $x \in X$ to its equivalence class $[x]$.

With this definition the map p becomes continuous. Moreover,

Question 1. Let $f : X \to Y$ be a map from X to some topological space Y such that $f(x_1) = f(x_2)$ whenever $x_1 \sim x_2$. We can then define a map $\bar{f} : X/\sim \to Y$ by $\bar{f}([x]) = f(x)$. Prove that f is continuous if and only if $\bar{f}$ is.

One source of examples comes from group actions. Namely, let X be a topological space, G a topological group and $G \times X \to X, (g, x) \to g \cdot x, g \in G, x \in X$ a continuous left action of G on X. Then for $x_1, x_2 \in X$ we set $x_1 \sim x_2$ iff there is a $g \in G$ such that $x_2 = g \cdot x_1$. In general the resulting space, which we will denote X/G, is not very well behaved: e.g., take $X = \mathbb{R}, G = \mathbb{R}^* = \mathbb{R} \setminus \{0\}$ with the standard product operation; G acts on X in a natural way and the quotient consists of two points, $[0]$ and $[1]$, the first of which is closed while the second is not.

Question 2. Show that if G is compact and X is Hausdorff then X/G is again Hausdorff.

Question 3. Set $X = \mathbb{R}^{n+1} \setminus \{0\}, G = \mathbb{R}^*$. The quotient space X/G will be denoted $\mathbb{R}P^n$. Show that it is compact and Hausdorff. Moreover, show that each point of $\mathbb{R}P^n$ has a neighbourhood homeomorphic to $\mathbb{R}^n$. [Hint: show that the natural map $S^n / \pm Id \to \mathbb{R}P^n$ is a homeomorphism.]

Question 4. More generally, set X to be the Stiefel manifold

$$V_m(\mathbb{R}^n) = \{(v_1, \ldots, v_m) \in \mathbb{R}^n \times \cdots \times \mathbb{R}^n \mid v_1, \ldots, v_m \text{ are linearly independent}\}.$$

We set $(v_1, \ldots, v_m) \sim (v'_1, \ldots, v'_m)$ iff $v_1, \ldots, v_m$ and $v'_1, \ldots, v'_m$ span the same m-plane. Show that $X/\sim$ is compact, Hausdorff, and each of its points has a neighbourhood homeomorphic to $\mathbb{R}^{m(n-m)}$. This space will be called the *Grassmannian of m-planes in* $\mathbb{R}^n$ and denoted $G_m(\mathbb{R}^n)$. [Hint: if $\mathbb{R}^n = V \oplus W$ then one can view V as the graph of some mapping $V \to W$, namely, the zero mapping. What other vector subspaces of $\mathbb{R}^n$ of dimension $\dim V$ can be regarded as graphs of linear mappings $V \to W$?]

In a similar way one defines the complex projective space $\mathbb{C}P^n$ and the quaternionic projective space $\mathbb{H}P^n$, and also complex and quaternionic Grassmannians, denoted $G_m(\mathbb{C}^n)$ and $G_m(\mathbb{H}^n)$ respectively. In the quaternionic case there are two options: one can consider either left or right quaternionic subspaces. This does not make much difference, as every left quaternionic subspace can be turned into a right one, and vice versa, via the quaternionic conjugation $a + bi + cj + dk \mapsto a - bi - cj - dk, a, b, c, d \in \mathbb{R}$, so the resulting quaternionic Grassmannians are homeomorphic.

Another source of examples is as follows. Suppose we have two topological spaces X, Y and a continuous map $f : X' \to Y$ where X' is a subspace of X. Then we can define an equivalence relation $\sim$ on $X \sqcup Y$ as follows: for any $x_1, x_2 \in X'$ such that $f(x_1) = f(x_2) = y \in Y$ we set $x_1 \sim x_2 \sim y$. The quotient space will be denoted as $X \cup_f Y$ and will be called the *result of attaching X to Y along f*. In the particular case when Y is a point we will write X/X' instead of $X \cup_f Y$.

Ths *mapping cylinder* $Cyl(f)$ of a continous map $f : X \to Y$ is defined as $(X \times [0, 1]) \cup_{f_0} Y$ where $f_0 : X \times \{0\} \to Y$ is given by $f_0(x, 0) = f(x)$. The space $Cone(f) = Cyl(f)/(X \times \{1\})$ will be called the *mapping cone* of f.

Question 5. (*) Show that the mapping cylinder of the natural map $S^n \to \mathbb{R}P^n$ is homeomorphic to $\mathbb{R}P^{n+1}$.

Homotopy

Two continuous maps $f, g : X \to Y$ are said to be *homotopic* if there is a continuous map $F : X \times [0, 1] \to Y$ such that for all $x \in X$ we have $F(x, 0) = f(x), F(x, 1) = g(x)$. We write $f \sim g$ to denote that f and g are homotopic.

Question 6. Prove that being homotopic is an equivalence relation on the set of all continuous maps $X \to Y$.

Two topological spaces X and Y are *homotopy equivalent* iff there are continuous maps $f : X \to Y, g : Y \to X$ such that $f \circ g \sim id_Y, g \circ f \sim id_X$.

Question 7. a) Let $X \subset \mathbb{R}^n$ be a set. Assume there is an $x \in X$ such that for all $x' \in X$ the segment that joins x and x' is included in X (note that this holds e.g. when X is convex). Show that X and $\{x\}$ are homotopy equivalent.

b) Set

$$X = S^{n-1} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid \sum x_i^2 = 1\}$$

and $Y = \mathbb{R}^n \setminus \{0\}$. Show that X and Y are homotopy equivalent.

Question 8. Prove that the mapping cylinder of a continuous map $f : X \to Y$ is homotopy equivalent to Y.

Homological algebra

Question 9. Let $0 \to C'_* \xrightarrow{i} C \xrightarrow{p} C''_* \to 0$ is a short exact sequence of complexes with differentials $\partial', \partial, \partial''$. Let $[c''] \in H_i(C''_*)$ be the homology class of a cycle $c'' \in C''_i$. The cycle c'' is equal $p(c), c \in C_i$ and $p(\partial c) \in C''_{i-1}$ is 0, as $p(\partial c) = \partial''(p(c)) = \partial'' c'' = 0$. So $\partial c = i(c')$ for some $c' \in C'_{i-1}$.

We have $\partial' c' = 0$, as $i(\partial' c') = \partial(i(c')) = \partial(\partial(c)) = 0$. Moreover, we saw in the lectures that if c' is constructed from $[c'']$ by the above procedure, then $[c''] \mapsto [c']$ gives a well-defined map $\triangle : H_i(C''_*) \to H_{i-1}(C'_*)$. This means that if $\tilde{c} \in C_i$ is a chain such that $p(\tilde{c}) = c'' + \partial'' \tilde{c}'', \tilde{c}'' \in C''_{i+1}$, then for the cycle $\tilde{c}' \in C'_{i-1}$ such that $i(\tilde{c}') = \partial \tilde{c}$ we have $[c'] = [\tilde{c}']$.

This easily implies that the map $\triangle : H_i(C''_*) \to H_{i-1}(C'_*)$ is a group homomorphism (lectures).

Prove that the sequence

$$\cdots \to H_i(C'_*) \to H_i(C_*) \to H_i(C''_*) \xrightarrow{\triangle} H_{i-1}(C'_*) \to \cdots$$

is exact (here the maps $H_i(C'_*) \to H_i(C_*)$ and $H_i(C_*) \to H_i(C''_*)$ are induced by i and p respectively, and $\triangle$ is the map that we have just constructed).

The following diagram might be of help.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C'_{i+1} & \longrightarrow & C_{i+1} & \longrightarrow & C''_{i+1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C'_{i} & \longrightarrow & C_{i} & \longrightarrow & C''_{i} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C'_{i-1} & \longrightarrow & C_{i-1} & \longrightarrow & C''_{i-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C'_{i-2} & \longrightarrow & C_{i-2} & \longrightarrow & C''_{i-2} & \longrightarrow & 0
\end{array}$$