Representations of $GL(2)$ over finite and local fields

This is a course for seniors and graduate students. The prerequisites include 2 years of algebra and analysis. This course cannot be taken without the research seminar "Automorphic representations of $GL(2)$".

The complex irreducible characters of finite groups $GL(2, \mathbb{F}_q)$ were computed by Dickson more than a hundred years ago. About 60 years ago this was generalized to $GL(n, \mathbb{F}_q)$ by Green. The resulting theory is a q-analogue of Frobenius' theory of characters of symmetric groups. It involves the famous combinatorial objects: the Hall-Littlewood and Kostka-Foulkes polynomials.

About 40 years ago Drinfeld realized that all the representations of $GL(2, \mathbb{F}_q)$ can be realized in the étale cohomology of certain curves over $\mathbb{F}_q$. The development of this geometric approach has led Lusztig to the realization of all the irreducible characters of $GL(n, \mathbb{F}_q)$ as Frobenius trace functions of some irreducible perverse Weil sheaves: character sheaves.

Coming back to $GL(2, \mathbb{F}_q)$, the following 50-year old observation goes back at least to Gelfand: the matrix elements of irreducible representations of this group provide the finite field analogs of the Bessel, Whittaker, hypergeometric and Γ-functions.

If we replace the finite field $\mathbb{F}_q$ with a local field $K = \mathbb{F}_q((t))$ or $\mathbb{Q}_p$, all the irreducible representations of $GL(2, K)$, except for 1-dimensional characters, become infinite-dimensional. However, their classification remains very similar to the case of finite field. In fact, according to the key observation of Gelfand, the dependence on K of the classification and values of the irreducible characters is essentially algebraic. Technically, one of the most efficient instruments in the representation theory of $GL(2, K)$ is A. Weil representation (of $Mp(4, K)$). For a finite field, the integral kernel of the Weil representation was described by Deligne as the Frobenius trace of a certain irreducible perverse sheaf about 30 years ago. The consequences for the Weil representation over local fields were realized only recently by Lafforgue and Lysenko.

If F is a global field, such as $\mathbb{F}_q(X_0)$ (rational functions on a curve over a finite field) or $\mathbb{Q}$, and $\mathbb{A}_F$ is its adele ring, then according to Langlands, the irreducible representations of $GL(2, \mathbb{A}_F)$ appearing in $L^2(GL(2, \mathbb{A}_F)/GL(2, F))$ (automorphic representations) are classified in terms of 2-dimensional representations of $\mathrm{Gal}(\overline{F} : F)$. In case of functional field F this was proved by Drinfeld; he developed the geometric representation theory as a tool for his proof. In case of rational numbers, only partial results are known. Among them, Deligne's proof of Ramanujan's conjecture (about the modular form Δ).