SAMPLES OF HOMEWORK AND EXAM PROBLEMS
FOR THE COURSE "RIEMANN SURFACES"

Elementary properties

1. Find zeroes and poles (with multiplicities) of the function $(1+z^2)/(1-z)$ on $\bar{\mathbb{C}} = \mathbb{CP}^1$ (2 points).

2. Put

$$X = \{(z, w) \mid z^3 + w^3 = 2\} \subset \mathbb{C}^2;$$

define a function $f\colon X \to \mathbb{C}$ by the formula $f\colon (z, w) \mapsto zw - 1$. Find $\operatorname{ord}_p(f)$, where $p = (1, 1)$. (3 points)

Find zeroes and poles (with multiplicities) for the following differential forms:

3. $z^2dz/(z + 1)$ on $\bar{\mathbb{C}} = \mathbb{CP}^1$ (2 points);

4. zdw on the Riemann surface

$$X = \{(z, w)\colon z^4 + w^4 = 1\}.$$

Riemann surfaces corresponding to algebraic equations

1. Construct a compact Riemann surface X corresponding to the equation $w^2 = (z - a_1)(z - a_2)(z - a_3)(z - a_4)$. Describe explicitly local coodinates at the points where z cannot serve as one.

2. Find poles (and their multiplicities) of the meromorphic form dw on the Riemann surface X.

3. Find genus of the compact Riemann surface corresponding to the equation $w^4 = z^3 - 1$.

4. Find poles of the form dz/w^4 on the above surface; at each of the poles, find residue of this form.

Branched coverings

1. Suppose that X is a compact Riemann surface and that $f\colon X \to \bar{\mathbb{C}}$ is a non-constant holomorphic mapping ramified over exactly two points of $\bar{\mathbb{C}}$. Prove that genus of X is zero and decribe the ramification of f.

2. Suppose that $f\colon X \to Y$ is a non-constant holomorphic mapping of compact and connected Riemann surfaces. Prove that

$$(\text{genus of } X) \geq (\text{genus of } Y).$$

3. Suppose that X is a compact Riemann surface of genus 3, Y is a compact Riemann surface of genus 2, and $f\colon X \to Y$ is a non-constant holomorphic mapping.

(a) Prove that f is unramified.

(b) What can be said about $\deg f$?

Divisors and linear systems

1. Put $X = \bar{\mathbb{C}}$; for the divisor

$$D = 2 \cdot 1 + 1 \cdot 3 - 1 \cdot \infty$$

on X, find a basis of the vector space $L(D)$.

2. Find a meromorphic form ω on $\bar{\mathbb{C}}$ such that $(\omega) = 1 \cdot 0 - 3 \cdot 1$.

3. Denote by X the hyperelliptic Riemann surface corresponding to the equation

$$w^2 = (z - a_1) \dots (z - a_6),$$

where $a_1, \dots, a_6$ are distinct complex numbers. Denote by $\pi \colon X \to \bar{\mathbb{C}}$ the standard projection. If $P \in X$ is the only element of $\pi^{-1}(a_1)$, find $l(nP)$ for each $n \in \mathbb{Z}$.

4. (a) Suppose that X is a compact Riemann surface of genus $g > 2$. Prove that the linear system $|2K_X|$ defines an embedding of X into a projective space. Find dimension of this projective space.

(b) What happens if $g = 2$?

5. Suppose that $X \subset \mathbb{CP}^2$ is a smooth projective curve of degree 4 and that P, Q, and R are three distinct points of X. Find $l(P + Q + R)$.

Elliptic curves

1. Express the EWisenstein series G_4 in terms of G_2 and G_3.

2. Find j-invariant of the elliptic curve $\mathbb{C}/\langle 1, i \rangle$.