HSE/Math in Moscow 2013-2014// Topology 1 // Revision problems for class test

Question 1. Let $f : X \to Y$ be a continuous map. For $x_1, x_2 \in X$ set $x_1 \sim x_2$ iff $f(x_1) = f(x_2)$. The map f is called a *quotient map* iff the induced map $\bar{f} : X/\sim \to f(X)$ is a homeomorphism.

(a) Give an example of a quotient map and of a non-quotient map.

(b) Show that f is a quotient map iff the following is true: $U \subset f(X)$ is open in the topology induced from Y iff $f^{-1}(U)$ is open in X.

(b) Show that if X is compact and Y is Hausdorff then any continuous map $f : X \to Y$ is a quotient map.

Question 2. A topological space X is *locally path connected* iff for every $x \in X$ and every open $U \ni x$ there is an open path connected subset V such that $U \supset V \ni x$.

(a) Give an example of path connected connected space which is not locally path connected.

(b) Show that the path components of a locally path connected space are open. Deduce that they are closed.

(c) Show that for a path connected space the connected components and the path components coincide.

Remark. In order for the conclusions of parts 2 and 3 to hold a weaker assumption than local path connectedness suffices: instead of requiring that for every $x \in X$ and *every* open $U \ni x$ there is an open path connected subset V such that $U \supset V \ni x$ it suffices to require that for every x there is *at least one* open path connected $U \ni x$.

Question 3. (a) Show that $GL_n(\mathbb{C})$ is path connected.

(b) Is $GL_n(\mathbb{R})$ path connected? If not then how many path components does it have and do they coincide with the connected components?

Question 4. Let X be a countably compact metric space. Recall that this means that every *countable* open cover of X has a connected subcover.

(a) Show that X is totally bounded. [Hint: suppose not; show that then there is an $\varepsilon > 0$ and countable family $A = \{x_1, x_2, \ldots\}$ of elements of X that are $\geq \varepsilon$ apart from one another; deduce that the union of $\varepsilon/2$-balls centred at the elements of A or any subset of it is closed and construct a countable open cover without a finite subcover.]

(b) Show that X is dense and deduce that it has a countable base.

(c) Deduce from (b) that for any open cover $\mathcal{U}$ of X there is a countable open cover $\mathcal{V}$ of X such that every $V \in \mathcal{V}$ is contained in some $U \in \mathcal{U}$. Deduce that X is compact.

Question 5. Let S be the surface obtained from the polygon shown on the blackboard.

(a) Calculate the number of the equivalence classes of vertices and deduce the Euler characteristic of S. State whether S is orientable and explain your answer.

(b) By cutting and pasting transform P into a polygon without adjacent edges of type 1 and such that all vertices are equivalent.

(c) By cutting and pasting again transform the resulting polygon into one with standard boundary.