HSE/Math in Moscow 2013-2014// Topology 1 // Problem sheet 7

Homotopy and homotopy equivalence continued

Question 1. Let X be the subspace of $\mathbb{R}^n$ given by

$$x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_n^2 = 1.$$

Show that X is homotopy equivalent to S^{p-1}, the intersection of X with the subspace of $\mathbb{R}^n$ given by $x_{p+1} = \cdots = x_n = 0$. [Hint: try to increase $x_1, \ldots, x_p$ while decreasing $x_{p+1}, \ldots, x_n$ and keeping $x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_n^2$ fixed.]

Question 2. Let X be the subspace of $\mathbb{C}^n$ given by

$$x_1^2 + \cdots + x_n^2 = 1.$$

Show that X is homotopy equivalent to S^{n-1}. [Hint: identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ and show that the homotopy constructed in question 1 for the hypersurface $\sum(\operatorname{Re} x_i)^2 - \sum(\operatorname{Im} x_i)^2 = 1$ preserves X.]

Suppose $\mathbb{R}^n$ is equipped with a positive definite bilinear form $(\cdot, \cdot)$. For any $v \in \mathbb{R}^n$ we define the projection map $pr_v : \mathbb{R}^n \to \mathbb{R}^n$ by

$$pr_v(w) = \frac{(w, v)}{(v, v)} v.$$

Given a basis $(e_1, \ldots, e_n)$ of $\mathbb{R}^n$ form a sequence $(f_1, \ldots, f_n)$ of vectors as follows:

$$f_1 = e_1, f_2 = e_2 - pr_{f_1}(e_2), f_3 = e_3 - pr_{f_1}(e_3) - pr_{f_2}(e_3), \ldots, f_n = e_n - \sum_{j=1}^{n-1} pr_{f_j}(e_n).$$

Question 3. (a) Check by induction that for all $i = 1, \ldots, n$

- the span of $e_1, \ldots, e_i$ coincides with the span of $f_1, \ldots, f_i$;
- $f_1, \ldots, f_i$ are pairwise orthogonal.

So $(f_1, \ldots, f_n)$ is an orthogonal basis of $\mathbb{R}^n$ and if we set $g_i = \frac{f_i}{\sqrt{(f_i, f_i)}}, i = 1, \ldots, n$ we get an orthonormal basis. In this way we construct an orthonormal basis out of a given basis; this algorithm is called the *Gram-Schmidt* process.

(b)Use the above to show that $O(n)$, the group of real orthogonal $n \times n$-matrices, is homotopy equivalent to $GL_n(\mathbb{R})$, the group of all real $n \times n$-matrices with non-zero determinant. [Hint: use the Gram-Schmid process to construct a map $GL_n(\mathbb{R}) \to O(n)$ and show that it is the identity on $O(n) \subset GL_n(\mathbb{R})$ and that it is homotopic to the identity viewed as a map $GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$.]

CW-complexes

Our next topic will be CW-complexes. This is a rather important class of topological spaces. In fact, it wouldn't be an overstatement to say that most spaces algebraic topologists care about are CW-complexes, or at least are homotopy equivalent to such. In the sequel we'll look into some basic properties of CW-complexes in some detail but first let us consider a few examples.

A *CW-complex* is a Hausdorff topological space X such that there are continuous maps $\gamma_\alpha^n : D^n \to X$ (where α just runs through some index set) with the following properties:

- The restriction of γ_α^n to the interior of D^n is injective; the image of the interior of D^n under γ_α^n is called *a(n open) cell of X*.
- X is the disjoint union of its open cells;
- (W) A subset of X is closed iff its intersection with the closure of each cell is closed.
- (C) The closure of any cell is contained in a union of finitely many cells.

Question 4. Show that if a topological space X is the union of finitely many closed subspaces $Y_1, \dots, Y_k$ of X, then a subset $Z \subset X$ is closed iff $Z \cup Y_i$ is closed for all i. Deduce that if X is Hausdorff and there are finitely many maps $\gamma_\alpha^n : D^n \to X$ which have the first two of the above properties, they will automatically have the other two properties (C and W).

So if we are able to construct finitely many maps from closed unit disks to X such that their restrictions to the interiors are injective and the images of the interiors cover the whole of X, we can conclude that X is a CW-complex.

Question 5. Show that the following spaces are CW-complexes.
(a) S^n [Hint: if you are stuck try to do this first for $n = 1$ and 2.]
(b) $\mathbb{R}P^n$. [Hint: again it might be a good idea to try $n = 1$ and 2 first.]
(c) The connected sum of g 2-tori. [Hint: use the classification of surfaces.]
(d) The connected sum of g real projective planes. [Hint: same as in part (c).]
(e) The connected sum of g 2-tori with n boundary components.
(f) The connected sum of g real projective planes with n boundary components.
That'll do for now, although we'll see more examples in the future. In particular, we'll see that CW-complexes can be quite complicated and they like to play tricks on unwary people.

Problems for discussion

Recall that the *genus* of $S = T^2 \# \cdots \# T^2$ is g (by definition). The genus and the Euler characteristic of S are related by $\chi(S) = 2 - 2g$. Similarly, the *genus* of $S = \mathbb{R}P^2 \# \cdots \# \mathbb{R}P^2$ is g and we have $\chi(S) = 2 - g$.

Let S be a compact surface S with boundary. We assume that $\mathrm{bd}(S)$ is homeomorphic to a disjoint union of circles. The *genus* of S is the genus of the surface without boundary obrained by attaching 2-disks to all boundary components of S.

1. Suppose a surface S is obtained from surfaces S_1 and S_2 by identifying a boundary component of S_1 with a boundary component of S_2. Show that $\chi(S) = \chi(S_1) + \chi(S_2)$. Deduce that if a surface S'' is obtained from a surface S' by removing a small round open disk from one of the polygons, then $\chi(S'') = \chi(S') - 1$.

2. Deduce that if S_1, S_2 are surfaces, then $\chi(S_1 \# S_2) = \chi(S_1) + \chi(S_2) - 2$ (we've seen a slightly different proof in the lectures).

3. Show that the genus of a surface with boundary is a topological invariant and calculate it in terms of the number of the boundary components and the Euler characteristic. Deduce that the genus, orientability and the number of boundary components form a complete set of invariants for compact surfaces.

4. (a) Let S' be a surface obtained from a surface S by attaching a thin ribbon to the boundary of S (see blackboard). Calculate $\chi(S')$ and the number of the boundary components of S' when the ribbon connects two different boundary components and when it connects two pieces of the same component.

(b) How are the genera of S and S' related, assuming both are orientable or both are non-orientable?

5. (a) Show that every surface with boundary can be obtained by attaching ribbons to the boundary of the unit disk $D^2 \subset \mathbb{R}^2 \subset \mathbb{R}^3$. Deduce that every surface with boundary can be embedded in $\mathbb{R}^3$ and every orientable surface with boundary can be immersed in $\mathbb{R}^2$.

6. Show that every surface without boundary can be obtained by taking the connected sum of $S^2, \mathbb{R}P^2$ or the Klein bottle and some number of 2-tori. Deduce that every orientable surface without boundary can be embedded in $\mathbb{R}^3$ and that all non-orientable surfaces can be immersed in $\mathbb{R}^3$ and embedded in $\mathbb{R}^4$, provided the Klein bottle and $\mathbb{R}P^2$ can be. We've seen an immersion of the Klein bottle in the lectures. Can you construct one for $\mathbb{R}P^2$?

A *triangulated* surface is a surface S obtained from a finite disjoint union $\bigsqcup \triangle_i$ of triangles in such a way that 1. no two distinct points of any $\triangle_i$ are glued together; 2. the intersection of the images in S of any two triangles is a single common edge, or a single common vertex, or empty.

7. (a) Show that for a triangulated surface S with empty boundary the number e of edges is $\frac{3}{2}t$ where t is the number of triangles, and express both e and t in terms of $\chi(S)$ and v, the number of vertices.

(b) Show that $e \leq \frac{v(v-1)}{2}$ and deduce that

$$v \geq \frac{1}{2}(7 + \sqrt{49 - 24\chi(S)}).$$

8. It follows from the previous question that a triangulation of S^2 can not have less than 4 vertices, while a triangulation of T^2 can't have less than 7. Can you construct a triangulation of the 2-torus with precisely 7 vertices?

Remark. In fact, it turns out that the above bound is sharp for almost all surfaces. Namely, a surface S admits a triangulation that has the smallest number of vertices allowed by the above bound, unless S is Klein's bottle ($\mathbb{R}P^2 \# \mathbb{R}P^2$), the non-orientable surface of genus 3 ($\mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2$) or the orientable surface of genus 2 ($T^2 \# T^2$).