HSE/Math in Moscow 2013-2014// Topology 1 // Problem sheet 8

CW-subcomplexes

A *CW-complex* is a Hausdorff topological space X such that there are continuous maps $\gamma^n_\alpha : D^n \to X$ (where α runs through some index set) with the following properties:

- The restriction of γ^n_α to the interior of D^n is **a homeomorphism onto its image**; the image of the interior of D^n under γ^n_α is denoted e^n_α and is called *a(n open) n-cell of dimension* n or *a(n open) n-cell* of X.
- X is the disjoint union of its open cells;
- (W) A subset of X is closed iff its intersection with the closure of each cell is closed.
- (C) The closure of any n-cell is contained in a union of finitely many cells **of dimension** $< n$.

A *CW-subcomplex* of a CW-complex X is a closed subspace of X that is a union of cells. If $Y \subset X$ is a CW-subcomplex of a CW-complex, then X/Y is again a CW-complex: let $p : X \to X/Y$ be the projection; whenever $e^n_\alpha \cap Y = \varnothing$ we define the map $\bar{\gamma}^n_\alpha$ as the composition $p \circ \gamma^n_\alpha$; these, together with the map that takes the point D^0 to what has become of Y, give one a CW-structure on X/Y.

Question 1. Let X be the 2-torus equipped with the CW-structure shown on the blackboard (one 0-cell, two 1-cells and one 2-cell. List all CW-subcomplexes of X and the corresponding quotient complexes.

Question 2. Give an example of a CW-complex such that the closure of an open cell is not a subcomplex.

Wedges

A *pointed space* is a couple (X, x) consisting of a topological space X and point $x \in X$, called the *basepoint*. Suppose we have a (possibly infinite) family $\{(X_i, y_i)\}$ of pointed spaces. Set $X = \bigsqcup X_i$ and let $\sim$ be the equivalence relation on X generated by $x_i \sim x_j, i \neq j$. The quotient $X/\sim$ will be called the *wedge* of $\{(X_i, y_i)\}$ and will be denoted $\bigvee_i (X_i, x_i)$. Whenever we need a basepoint in this space, we take it to be the equivalence class of all x_i's. When it is clear or irrelevant which points $x_i \in X_i$ we take, we'll be dropping them and simply writing X_i, $\bigvee_i X_i$ etc.

It follows from the definition of quotient topology that if Y is another topological space then a continuous map $f : \bigvee_i (X_i, x_i) \to Y$ is the same as a family $f_i : X_i \to Y$ of continuous maps such that $f_i(x_i) = f_j(x_j)$ for all i, j.

Question 3. Let X be a CW-complex such that it has one 0-cell and all its other cells have the same dimension $n > 0$. Show that X is homeomorphic to a wedge of n-spheres. [Hint: show that if $\bigvee_i (X_i, x_i)$ is the wedge of $\{(X_i, y_i)\}$ then a neighbourhood of the basepoint is the $\bigvee_i (U_i, x_i)$ where U_i is an open subset of X_i that contains x_i.]

Question 4. Show that, conversely, a wedge of spheres of dimensions > 0 (those dimensions do not have to be the same) can be equipped with a CW-structure with one 0-cell and some number of cells of dimension > 0, one per sphere in the wedge.

Question 5. Show that an infinite wedge of circles is not metrisable, i.e., that the wedge topology on it can not be induced by any metric. [Hint: take any metric that induces the usual topology on each circle and for any $\varepsilon > 0$ construct an open subset that contains the basepoint x but not contained in $U(x, \varepsilon)$.]

So some even relatively simple CW-complexes are too large to live in $\mathbb{R}^n$ (or, in fact, any metric space). The reason is that they contain a lot of open sets, which is precisely why they often come in useful: there are lots of continuous maps from them to other spaces.

Question 6. For each of the following spaces decomposed as unions of cells state which of the properties (C), (W) hold, if any.

(a) The unit disk in $\mathbb{R}^2$ represented as the union of its interior and the elements of the boundary considered as 0-cells.

(b) Let C_n be the circle in $\mathbb{R}^2$ with centre at $(\frac{1}{n}, 0)$ and of radius $\frac{1}{n}$. The union $\bigcup_n C_n$ is called the *Hawaiian earring*. It can be represented as the union of the origin $(0, 0)$, which is the 0-cell, and the spaces $C_n \setminus \{(0, 0)\}$, each of which is homeomorphic to an open interval.

Part (b) illustrates what happens when one tries to fit an infinite wedge into a Euclidean space: one ends up with a space that looks as if it is homeomorphic to the wedge but in fact isn't.

Connectedness

Recall that the *n-skeleton* X^n of a CW-complex X is the union of all cells of X of dimenion $\leq n$. This is a CW-subcomplex of X.

Question 7. Show that for a CW-complex X the following properties are equivalent:
(i) X is path-connected.
(ii) X is connected.
(iii) X^1 is connected.
(iv) X^1 is path connected.
[Hint: one could try showing that 1. every point can be joined with an element of X^0 and 2. that every element of X^1 has a path connected neighbourhood, and then apply one of the results from the revision sheet.]

Problems for discussion

Earlier in this course we have considered applications of topology to algebra ("the main theorem of algebra") and analysis (existence and uniqueness of solutions of ODE's). Let us now consider an application to algebraic geometry. A *complex plane projective curve* is the set of all $(x_0 : x_1 : x_2) \in \mathbb{C}P^2$ such that $f(x_0 : x_1 : x_2) = 0$ where f is a homogeneous polynomial of degree d with complex coefficients. Such a curve is *non-singular* iff the gradient of f does not vanish at any point of the curve. It can be shown that if this is the case, then the curve is homeomorphic to a compact orientable surface of genus $g = \frac{(d-1)(d-2)}{2}$ with empty boundary. We do not show this here but hopefully will show this later, when we've seen the Riemann-Hurwitz formula.

Real plane projective curves are defined in a similar way. **Harnack's theorem** says that a non-singular (in the complex sense) degree d plane projective curve in $\mathbb{R}P^2$ can have no more than $\frac{(d-1)(d-2)}{2} + 1$ connected components ("ovals"). Below we give a proof of that, modulo some facts from differential geometry; these facts seem intuitvely plausible but their proof requires a bit of machinery which we don't have, so we do not attempt it here.

Below X is a smooth compact orientable surface X, possibly non-connected and possibly with boundary. Recall that the *genus* $g(X)$ of X is the sum of the genera of all connected surfaces that are obtained by attaching disks to all boundary components of X.

1. Let $C \subset X$ be a curve that does not intersect the boundary and let Y be the result of *cutting* X *along* C, i.e. Y is X minus a small open tubular neighbourhood of C. Show that $g(Y) - b_0(Y) = g(X) - b_0(X) - 1$ where b_0 denotes the 0-th Betti number, i.e., the number of the connected components.

2. Now suppose X be a closed genus g surface. (Recall that *closed* means connected, orientable and without boundary.) Deduce from the previous part that if we cut X along $g+2$ pairwise non-intersecting curves, the resulting surface will have at least 3 connected components.

3. Let X be as in the previous question and let σ be a smooth *involution* of X, i.e., σ is a smooth map $X \to X$ such that $\sigma \circ \sigma = \mathrm{Id}$. Let us denote the fixed point set of X as X^σ. Assume that X^σ is a union of pairwise non-intersecting curves. Then it can be shown that $Y = X/\langle\sigma\rangle$ can be made into a smooth surface with boundary homeomorphic to X^σ, so that the natural quotient map $X \to Y$ is smooth. We will assume this. Show that Y is connected and deduce that so is $(X \setminus X^\sigma)/\langle\sigma\rangle$.

4. Deduce from the previous question that $X \setminus X^\sigma$ has at most two boundary components. Deduce using question 2 that the number of the components of X^σ is at most $g+1$.