HSE/Math in Moscow 2013-2014// Topology 1 // Problem sheet 1

Topological spaces and continuous maps

Recall that a *topological space* is a couple $(X,\mathcal{U})$ where X is a set and $\mathcal{U}$ is a collection of subsets of X such that

- $\varnothing, X \in \mathcal{U}$,
- $\mathcal{U}$ is closed under taking *finite* intersections and *arbitrary* unions.

If such a collection $\mathcal{U}$ is given it is called a *topology* on X and its elements are called *open subsets*. Subsets of X of the form $X \setminus U, U \in \mathcal{U}$ are called *closed*. Finite unions and arbitrary intersections of closed subsets are again closed. In the sequel, when it is clear which topology on X is meant we will simply write X instead of $(X,\mathcal{U})$.

Example. One of the most important topological spaces is the set $\mathbb{R}$ of real numbers equipped with the following topology: a set $X \subset \mathbb{R}$ is open iff for any $x \in X$ there are $a, b \in \mathbb{R}$ such that $x \in (a,b) \subset X$. The Euclidean space $\mathbb{R}^n$ can be made into a topological space in a similar way: a set $X \subset \mathbb{R}^n$ is open iff for any $x = (x_1, \ldots, x_n) \in X$ there are $a_1, b_1, \ldots, a_n, b_n \in \mathbb{R}$ such that $x \in \Pi_{i=1}^n (a_i, b_i) \subset X$. (Here $\Pi_{i=1}^n (a_i, b_i)$ stands for the set of all $(t_1, \ldots, t_n) \in \mathbb{R}^n$ such that $a_i < t_i < b_i$ for all $i = 1, \ldots, n$; this is the Cartesian product of the intervals (a_i, b_i).)

If X is a topological space and $Y \subset X$ then we can introduce *the induced topology* on Y by declaring a set $U \subset Y$ open iff $U = Y \cap V$ where V is open in X. Note that with this definition the inclusion $Y \subset X$ becomes continuous. We will often refer to Y with the induced topology as a *subspace* of X. So, starting from the previous example we can construct many more. In particular, we get topologies on $(0,1)$, $[0,1]$ and $[0,1)$.

Question 1. a) Let X be a topological space, $U \subset X$ an open subset and $V \subset U$ a subset that is open in the induced topology. Prove that V is open in X.

b) Let X be a topological space, $Z \subset X$ a closed subset and $W \subset Z$ a subset that is closed in the induced topology. Prove that W is closed in X.

A map $f : X \to Y$ of topological spaces is called *continuous* if one of the following equivalent conditions is satisfied.

1. If $U \subset Y$ is open, than $f^{-1}(U)$ is open in X.
2. If $Z \subset Y$ is closed, than $f^{-1}(Z)$ is closed in X.
3. For any $x \in X$ and any open $U \subset Y$ that contains $f(x)$ there is an open set $V \subset X$ such that $x \in V$ and $f(V) \subset U$.

If the third condition is satisfied for some $x \in X$ then we say that f is *continuous at* x. So $f : X \to Y$ is continuous if and only if it is continuous at every $x \in X$. Note that in the case $X \subset Y = \mathbb{R}$ the third condition is precisely the definition of continuity that can be found in most undergraduate analysis books.

In practice it is often convenient to glue continuous maps from maps defined on subspaces. The following shows when this is possible.

Question 2. a) Suppose a topological space X is represented as a union $X = \bigcup_{i \in I} X_i$ of its subspaces. Let $f : X \to Y$ be a map from X to another topological space Y. Suppose $f|X_i : X_i \to Y$ is continuous for all $i \in I$. Prove that then f is itself continous in each of the following cases:

(i) all X_i are open;

(ii) all X_i are closed and there are only finitely many of them (i.e., $\#I < \infty$).

b) Let $f : X \to Y$ be a continuous map such that $f(X)$ is included in some subset $Y' \subset Y$. Show that the resulting map $f : X \to Y'$ is continuous.

In algebra it often happens that if a bijection preserves some structure, so does the inverse bijection. E.g., if $f : G_1 \to G_2$ is a bijective group homomorphism, then $f^{-1} : G_2 \to G_1$ is again a group homomorphism. In topology this often fails to be true.

Question 3. Define $f : [0,1) \to S^1$ by $f(x) = (\cos 2\pi x, \sin 2\pi x)$; here $S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ with the topology induced from $\mathbb{R}^2$. We assume that f is continuous and bijective.

a) Show that $f^{-1} : S^1 \to [0,1)$ is not continuous.

b) For each of the maps f, f^{-1} state whether it is open and whether it is closed.

Notice that this example also shows that the topology on $f(X)$ can be different from the one on X, even if f is injective: in general, X contains "more" open sets than $f(X)$. One could again compare this to what happens in algebra: if $f : G_1 \to G_2$ is an injective group homomorphism, then G_1 and G_2 are isomorphic as groups.

Let X and Y be topological spaces. A continuous bijection $f : X \to Y$ is called a *homeomorphism* if f^{-1} is continuous. If such a map exists we say that X and Y are *homeomorphic.*

Question 4. a) Show that $(0, 1), (0, \infty), \mathbb{R}$ are all homeomorphic.

b) Show that $[0, 1)$ and $[0, \infty)$ are homeomorphic.

Connectedness

A topological space X is said to be

- *pathwise connected* if for all $x, y \in X$ there is a continuous map $f : [0, 1] \to X$ such that $f(0) = x, f(1) = y$.
- *connected* if for any two open $U, V \subset X$ such that $U \cap V = \varnothing$ and $U \cup V = X$ we have either $U = X$ and $V = \varnothing$, or vice versa.

Intermediate Value Theorem: If $a, b \in \mathbb{R}$ and $f : [a, b] \to \mathbb{R}$ is a continuous function, then for all $y \in [min(f(a), f(b)), max(f(a), f(b))]$ there is an $x \in [a, b]$ such that $f(x) = y$.

We will assume the IVT; it is not too hard to prove it using one's favourite completeness principle of the real numbers.

Question 5. a) Using the IVT show that $[0, 1]$ is connected. Deduce from this that any pathwise connected space is connected.

b) Let $X \subset \mathbb{R}^2$ be the union of the segment joining the points $x = 0, y = -1$ and $x = 0, y = 1$, and the graph of $f(x) = \sin \frac{1}{x}$ for $x \in (0, 1)$. Using the IVT show that X is not path connected.

Question 6. Show that for any $n > 1$ the spaces $(0, 1), [0, 1), [0, 1], \mathbb{R}^n, n > 1$ are pairwise non-homeomorphic.

Closures and interiors

Let X be a topological space. The *closure* $\bar{A}$ of a subset $A \subset X$ is the intersection of all closed subsets of X that contain A. The *interior* $Int(A)$ of $A \subset X$ is the union of all open subsets of X contained in A.

A subset A of a topological space X is called *dense* iff $\bar{A} = X$. A typical example would be $A = \mathbb{Q}$ inside $X = \mathbb{R}$. In the lectures we have shown that A is dense iff every open subset of X intersects A.

Question 7. a) Show that if a dense subset A of a topological space X is connected (in the induced topology), then X itself is connected.

b) Deduce from part (a) that the space from 5 b) is connected. So the converse of 5 a) does not hold.

Problems for discussion

Set

$$D^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid \sum x_i^2 \leq 1\}, S^{n-1} = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid \sum x_i^2 = 1\}.$$

If X is a topological space and $Y \subset X$ is a subspace, then a continuous map $f : X \to Y$ that is the identity on Y is called a *retraction.*

1. Brouwer's theorem says that every continuous self-map of D^n has a fixed point. Deduce Brouwer's theorem from the fact that there is no retraction $D^n \to S^{n-1}$. (The latter fact is most easily proved using homology theory; for $n = 2$ we'll give an alternative proof in the next problem sheet, or the one after that.)

2. Let $X = [-1, 1] \times [-1, 1]$ be a square in the plane $\mathbb{R}^2$ and let $\gamma_1, \gamma_2 : [0, 1] \to X$ be paths from $(-1, 1)$ to $(1, -1)$ and from $(1, 1)$ to $(-1, -1)$ respectively. Show that the images of γ_1 and γ_2 intersect.

3. There are two roads from town A to town B. Two people can travel from A to B, one taking one road and the other the other road, so that they start and arrive simultaneously and at each time the distance between them is ≤ 1 kilometer. Is is possible for two people to travel one from A to B, the other from B to A, so that again they start and arrive simultaneously and at each time the distance between them is > 1 kilometer?

A *graph* Γ is a set $V(\Gamma)$ of the *vertices* of Γ and another set $E(\Gamma)$ of the *edges* of Γ together with a map ev from $E(\Gamma)$ to the set of non-empty subsets of $V(\Gamma)$ with $\leq$ 2-elements. In other words, the edges are non-oriented, each edge either has two (unordered) endpoints, or has just one endpoint, in which case the edge is called a *loop.* Moreover, there may be several edges whose endpoints coincide; these are called *multiple* edges. In the sequel we assume that $V(\Gamma), E(\Gamma)$ are finite.

An *embedding* $i : \Gamma \to \mathbb{R}^2$ is an injective map $V(\Gamma) \to \mathbb{R}^2$ together with a choice, for each edge, of a piecewise linear path that joins the images of the endpoints; we assume that the paths that correspond to any two distinct edges do not intersect, except at the common endpoints (if any). The union of all these paths will be denoted $i(\Gamma)$. A graph is called *planar* if it admits an embedding into the plane.

4. Show that the graphs $K_{3,3}$ and K_5 (see blackboard) are not planar. (Conversely, if a graph is not planar, it has a subgraph isomorphic to $K_{3,3}$ or K_5. This is Kuratowski's theorem. It has several proofs; they are elementary but a bit long and tedious.)

5. Euler's formula. Show that if $i : \Gamma \to \mathbb{R}^2$ is an embedding, then $f = 1 + s - v + e$ where f is the number of the connected components of $\mathbb{R}^2 \setminus i(\Gamma)$, s is the number of the connected components of $i(\Gamma)$, e is the number of the edges and v the number of the vertices of Γ.

6. Five colour theorem. Using Euler's formula show that a planar graph without multiple edges has a vertex where ≤ 5 edges meet. Deduce that one can colour a planar graph using ≤ 5 colours so that the endpoints of every edge have different colours. (In fact, 4 colours suffice: this is the (in)famous four colour theorem. The first proof of it involved about a thousand pages of computer output. This has since been reduced to a few dozen pages but still those calculations, I believe, have never been checked by a human.)