HSE/Math in Moscow 2013-2014// Topology 1 // Problem sheet 10

Contractible spaces

Question 1. Question 9 on p. 19, A. Hatcher, Algebraic topology (English edition).

The fundamental group

Let $x_0 \in X$ be a point of a path connected space X. Continuous maps $\gamma : [0,1] \to X$ such that $\gamma(0) = \gamma(1) = x_0$ will be called *loops in X based at x_0* and the set of all such maps will be denoted $\Omega(X, x_0)$ or simply $\Omega(X)$ when it is clear or irrelevant which point we take as x_0. We introduce an equivalence relation $\sim$ on $\Omega(X, x_0)$ by declaring $\gamma_0 \sim \gamma_1$ iff there is a homotopy $F : [0,1] \times [0,1] \to X$ such that $F(s,0) = \gamma_0(s), F(s,1) = \gamma_1(s)$ for all $s \in [0,1]$ and $F(0,t) = F(1,t) = x_0$. Note that this is just a homotopy as defined in problem sheet 6 but this time it is only allowed to pass through loops based at x_0, and not just any maps $[0,1] \to X$.

Question 2. Show that this is indeed an equivalence relation.

The quotient $\Omega(X, x_0)/\sim$ will be denoted $\pi_1(X, x_0)$ and will be called the *fundamental group* of X. The reason it is called a group will be revealed shortly.

Recall that a *binary operation* on a set X is a map $* : X \times X \to X$. It is common to denote the result of applying $*$ to a couple (x, y) as $x*y$ rather than $*(x, y)$. For example, the usual arithmetic operations $+, -, \cdot$ are binary operations on the sets of integer, rational, real or complex numbers. Division is a bit trickier. E.g., it is a binary operation on $\mathbb{Q}^*$, the set of all *non-zero* rational numbers, but it is *not* a binary operation on $\mathbb{Q}$, as division by zero is not defined.

A set G with a binary operation $* : G \times G \to G$ is called a *group* iff the following axioms hold

- Associativity: $a * (b * c) = (a * b) * c$ for all $a, b, c \in G$.
- Existence of a unit: there is an $e \in G$ called a *unit element*, or simply a *unit*, such that $e * a = a * e = a$ for all $a \in G$.
- Existence of inverses: for any $a \in G$ there is an a^{-1} called the *inverse* of a such that $a^{-1} * a = a * a^{-1} = e$.

Here is a warm-up question on groups.

Question 3. Determine which of the following (if any) are groups, state which of the axioms hold (if any) and explain your answers.

(a) The integers under subtraction.
(b) The integers under multiplication.
(c) All self-maps of a given set under composition of maps.
(d) The residues 2,4,6,8 modulo 10 under multiplication.

If we have a binary operation $*$ on a set G, it is common to simply write gh instead of $g * h$. If the operation we consider is associative, then there is no need to write the brackets either, as, e.g., any way to put the brackets in $g_1g_2g_3g_4$ will result in the same element of G. Note that however, the order matters: gh needn't be equal hg.

Now let (X, x_0) again be a pointed topological space and let us define a binary operation on $\Omega(X, x_0)$: for $\gamma_0, \gamma_1 \in \Omega(X, x_0)$ set $\gamma_0 \cdot \gamma_1$ to be the loop γ such that $\gamma(t) = \gamma_0(2t)$ for $t \in [0, \frac{1}{2}]$ and $\gamma(t) = \gamma_1(2t-1)$ for $t \in [\frac{1}{2}, 1]$.

Question 4. Take $X = S^1$ and let $x_0 \in X$ be any point. (a) Show that the operation $\cdot$ defined above is not associative and does not have a unit. (b) Show however that if $\gamma_0 \sim \gamma_0'$ and $\gamma_1 \sim \gamma_1'$, then $\gamma_0 \cdot \gamma_1 \sim \gamma_0' \cdot \gamma_1'$.

Question 5. Using part (b) of the previous question we can define a binary operation on $\pi_1(X, x_0)$ as follows: $[\gamma_0] \cdot [\gamma_1] = [\gamma_0 \cdot \gamma_1]$. Show that this makes $\pi_1(X, x_0)$ a group.

Subgroups

A *subgroup* of a group G is a subset of G that is a group under the operation of G. A subgroup $H \subset G$ is *normal* iff for all $g \in G$

$$g^{-1}Hg = \{g^{-1}hg \mid h \in H\} = H.$$

Question 6. (a) Show that $\mathbb{Z}$ is a group under addition. Is $\mathbb{Z}_{\geq 0} \subset \mathbb{Z}$ a subgroup of $\mathbb{Z}$?

(b) Show that all self-*bijections* of a given set X form a group under composition of maps (cf. question 3 a). If $X = \{1, 2, \ldots, n\}$ then this group is denoted S_n and is called the *permutation group* on n letters. Give an example of a normal and a non-normal subgroup of S_3.

If $H \subset G$ is a subgroup and $g \in G$ then we define $gH = \{gh \mid h \in H\}$, the *left coset* of H in G, and $Hg = \{hg \mid h \in H\}$, the *right coset* of H in G.

Question 7. Let G be a group, $H \subset G$ a subgroup and g_1, g_2 elements of G.

(a) Show that the cosets $g_1 H$ and $g_2 H$ either are equal or do not intersect.

(b) Show that there is a bijection $g_1 H \to g_2 H$.

(c) Show that any element of G belongs to some left coset.

(d) Deduce Lagrange's theorem: the number of elements of any finite group is divisible by the number of elements of any of its subgroups.

So in particular, a group with 20 elements can't have a subgroup with 19 elements.

Problems for discussion

We keep the notation from the previous problem sheet.

1. Set H^i to be the half space

$$\{(x_1, \dots, x_i, 0, \dots, 0) \in \mathbb{R}^n \mid x_i > 0\}.$$

If σ is a sequence $1 \le \sigma_1 < \dots < \sigma_m \le n$ then set e'_σ to be the set of all orthogonal m-frames $(e_1, \dots, e_m)$ such that $e_i \in H^{\sigma_i}$. Show that any m-plane $\in e_\sigma$ has a unique basis $\in e'_\sigma$.

2. Our next task will be to show that the closure $\bar{e}'_\sigma$ in the space of all orthogonal m-frames (which, in turn, inherits a topology from the space of all $m \times n$ matrices) is homeomorphic to the closed unit disk of dimension $d(\sigma) = (\sigma_1 - 1) + (\sigma_2 - 2) + \dots + (\sigma_m - m)$. We proceed by induction on m. Show that the statement is true in the base case $m = 1$.

3. Assuming the statement from the previous question is true for m show it holds for $m + 1$. One could proceed as follows. Take a σ as above and a $\sigma_{m+1} > \sigma_m$. Let $v_i, i = 1, \dots, m$ be the σ_i-th vector of the standard basis of $\mathbb{R}^n$. For each frame $w = (w_1, \dots, w_m) \in \bar{e}'_\sigma$ construct using question 4 from the previous problem sheet, discussion part, a rotation T_w that takes each v_i to w_i and depends continuously on w. Let D be the subset of $\bar{H}^{\sigma_{m+1}}$ formed all unit vectors u that are orthogonal to each $v_1, \dots, v_m$. Show that D is homeomorphic to $D^{\sigma_{m+1} - m - 1}$ and that the maps

$$(w_1, \dots, w_m, u) \mapsto (w_1, \dots, w_m, T_w u), (w_1, \dots, w_m, w_{m+1}) \mapsto (w_1, \dots, w_m, T_w^{-1} w_{m+1})$$

$\bar{e}'_\sigma \times D \to \bar{e}'_{\sigma, \sigma_{m+1}}$ and $\bar{e}'_{\sigma, \sigma_{m+1}} \to \bar{e}'_\sigma \times D$ respectively are homeomorphisms inverse to one another where $\bar{e}'_{\sigma, \sigma_{m+1}}$ is the closure of $e'_{(\sigma_1, \dots, \sigma_m, \sigma_{m+1})}$.

4. Show that the natural map from m-frames to m-planes takes an element of $\bar{e}'_\sigma \setminus e'_\sigma$ to an element of $\bar{e}_\sigma \setminus e_\sigma$ and deduce that e_σ's form a cell decomposition of $G_m(\mathbb{R}^n)$.