HSE/Math in Moscow 2013-2014// Topology 1 // Problem sheet 9

CW-subcomplexes

A *CW-complex* is a Hausdorff topological space X such that there are continuous maps $\gamma_\alpha^n : D^n \to X$ (where α runs through some index set) with the following properties:

- The restriction of γ_α^n to the interior of D^n is **a homeomorphism onto its image**; the image of the interior of D^n under γ_α^n is denoted e_α^n and is called *a(n open) n-cell of dimension* n or *a(n open) n-cell* of X.
- X is the disjoint union of its open cells;
- (W) A subset of X is closed iff its intersection with the closure of each cell is closed.
- (C) The closure of any n-cell minus the cell itself is contained in a union of finitely many cells **of dimension** $< n$.

A *CW-subcomplex* of a CW-complex X is a closed subspace of X that is a union of cells. If $Y \subset X$ is a CW-subcomplex of a CW-complex, then X/Y is again a CW-complex: let $p : X \to X/Y$ be the projection; whenever $e_\alpha^n \cap Y = \varnothing$ we define the map $\bar{\gamma}_\alpha^n$ as the composition $p \circ \gamma_\alpha^n$; these, together with the map that takes the point D^0 to what has become of Y, give one a CW-structure on X/Y.

Question 1. Using the basic properties of CW-complexes shown in the lectures prove that a CW-subcomplex Y of a CW-complex X is again a CW-complex. [Hint for W: if we take a cell $e \subset X \setminus Y$ then its closure intersected with Y is contained in the union of the closures of finitely many celle in Y.]

Question 2. Again using the basic properties show that if Y is a CW-subcomplex of a CW-complex X, then X/Y is again a CW-complex with the maps $\bar{\gamma}_\alpha^n : D^n \to X/Y$ defined as above. You may assume that X/Y is Hausdorff.

Retracts

Let Y be a subspace of a topological space X. A *retraction* of X onto Y is a continuous map $f : X \to Y$ such that $f(y) = y$ for all $y \in Y$. If such a map exists we say that Y is a *retract* of X. If there is a homotopy $F : X \times I \to X$ such that $F(x, 0) = x, F(x, 1) \in Y$ for all $x \in X$ and $F(y, t) \in Y$ for all $y \in Y$ then we say that Y is a *weak deformation retract* of X and if, in addition to all the above $F(y, t) = y$ for all t, we say that X is a *strong deformation retract* of X. Strong deformation retracts are retracts, the retraction being given by $x \mapsto F(x, 1)$.

A continuous map $X \to Y$ with Y path connected is called *nullhomotopic* or *contractible* iff it is homotopic to a constant map. A topological space X is *contractible* iff it is homotopy equivalent to a one-point space.

We have seen in problem sheet 1 that S^{n-1} is not a retract of D^n, and we have subsequently shown this for $n = 2$ in problem sheet 2. A wedge $S^1 \vee S^1$ of two circles is a retract of any of the circles (this is clear) but is not a deformation retract (weak or strong) of either of them (we'll see this later). Finally, most deformation retracts that occur in everyday life are strong deformation retracts (we'll prove this for CW-subcomplexes of CW-complexes) but, as the example below shows, this is not always the case.

Question 3. Question 6 on p. 18, A. Hatcher, Algebraic topology (English edition).

Question 4. Question 9, ibid., p. 19. Note that, assuming S^{n-1} is not contractible, this shows that there is no retraction $S^{n-1} \to D^n$.

Homotopy extension property

Question 5. (a) Using the homotopy extension property show that (a) S^2 with three points identified and (b) two copies of S^2 with the North pole of one identified with the North pole of the other, and similarly for the South poles, are homotopy equivalent to a wedge of 2-spheres and circles. In each case determine the number of the spheres and the circles in the wedge.

Question 6. Study the proof of Corollary 0.21 on pp. 16-17 of Hatcher's Algebraic Topology and show that a continuous map $f : X \to Y$ of CW-complexes is a homotopy equivalence iff the cone of f is contractible.

Problems for discussion

Earlier in this course we have seen the Grassmann manifolds $G_m(\mathbb{R}^n)$. Here we would like to construct a CW-structure on them. This structure is called the *Schubert decomposition.*

1. Show that $G_m(\mathbb{R}^n)$ is homeomorphic to the quotient of the space of all real matrices with n columns and m rows of maximal rank by the left action of $GL_m(\mathbb{R})$.

2. Let $1 \le \sigma_1 < \cdots < \sigma_m \le n$ be a sequence of integers. Set e_σ to be the set of all m-planes V such that $\dim(V \cap \mathbb{R}^{\sigma_i}) = i, \dim(V \cap \mathbb{R}^{\sigma_i - 1}) = i - 1$. Show that $V \in e_\sigma$ iff it has a basis as shown on the blackboard.

3. Show that a basis of V of the form shown on the blackboard is unique and deduce that there is a continuous bijection $\mathbb{R}^d \to e_\sigma$ where $d = (\sigma_1 - 1) + (\sigma_2 - 2) + \cdots + (\sigma_m - m)$.

4. Suppose $(v_1, \ldots, v_m)$ and $(w_1, \ldots, w_m)$ are orthogonal m-frames such that $v_i \neq -w_i, i = 1, \ldots, m$ and $(v_i, w_j) = 0$ for $i > j$. Show that there is an element of $O_n(\mathbb{R})$ that takes each v_i to w_i, depends continuously on $(v_1, \ldots, v_m)$ and $(w_1, \ldots, w_m)$ and is the identity on the orthogonal complement of the span of $v_1, \ldots, v_m, w_1, \ldots, w_m$.