HSE/Math in Moscow 2013-2014// Topology 1 // Problem sheet 11

Groups

Question 1. Let H be a subgroup of a group G. Show that the following statements are equivalent.
(i) H is normal.
(ii) The left and right cosets of H coincide.
(iii) For each $g \in G$ and $h \in H$ we have $ghg^{-1} \in H$.

For a normal subgroup $H \subset G$ we can define a binary operation on the set of all (left or right) cosets: we set $g_1H \cdot g_2H = g_1g_2H$.

Question 2. (a) Show that this operation is well defined, e.g., if one takes $g_1' \in g_1H$ and $g_2' \in g_2H$ then $g_1'g_2'$ will be in g_1g_2H.
(b) Show by example that if H is not normal, then the conclusion of part (a) need not hold.
(c) Show that, assuming again that H is normal, the set of all H-cosets is itself a group under the above operation.

Let G, H be groups. A *group homomorphism* from G to H is a map $f : G \to H$ such that $f(g_1g_2) = f(g_1)f(g_2)$ for all $g_1, g_2 \in G$. A bijective group homomorphism is called an *isomorphism.* If there is an isomorphism $G \to H$ we say that G and H are *isomorphic* and we write $G \cong H$.

Question 3. Let $f : G \to H$ be a group homomorphism
(a) Show that if f is an isomorphism, then so is f^{-1}. So phenomena like continuous bijections whose inverses are not continuous, which cause quite a bit of trouble in topology, are non-existent in group theory.
(b) Show that the *kernel* $\ker f = f^{-1}(e)$ of f is a normal subgroup and construct, provided f is surjective, an isomorphism $G/\ker f \cong H$.

One way to rephrase the statement of part (b) would be to say that the homomorphic image of a group is isomorphic to the quotient by the kernel. While this statement is more or less obvious, and its proof straightforward, it may sound a bit esoteric to those who haven't studied group theory.

Let A be a set. The *free group* $F(A)$ *on the elements of* A is the set of all finite sequences $a_1^{\varepsilon_1}a_2^{\varepsilon_2}\cdots a_n^{\varepsilon_n}$ where $a_1, \ldots, a_n \in A$ and $\varepsilon_1, \ldots, \varepsilon_n \in \{\pm 1\}$, quotiented by the equivalence relation generated by $s_1aa^{-1}s_2 = s_1s_2$ and $s_1a^{-1}as_2 = s_1s_2$ where $a \in A$ and s_1, s_2 are sequences as above. We allow the empty sequence as well and we denote it e. We write a_i instead of a_i^1 and we abbreviate several *consecutive* occurrences of a_i, resp. a_i^{-1} as a_i^k, resp. a_i^{-k}.

Question 3. (a) Show that $F(A)$ is indeed a group with e as the unit element.
(b) We say that a subset S of a group G *generates* G iff the smallest subgroup of G containing S is G itself. If this is the case, show that any map of sets $A \to S$ extends to a group homomorphism. Deduce that if $f : A \to S$ is a surjective map then there if a surjective group homomorphism $\bar{f} : F(A) \to G$.

The *normal closure* $\langle S^G \rangle$ of a subset S of a group G is the smallest *normal* subgroup of G that contains S. This may coincide with the subgroup generated by S but usually does not. If we are given a surjective homomorphism $f : F(A) \to G$ and a subset $S \subset F(A)$ such that $\langle S^G \rangle = \ker f$ then we say that G is generated by the elements of A, subject to the set S of relations and we write $G \cong \langle A \mid S \rangle$. Sometimes one lists the elements of A and S explicitly. E.g., if $A = \{a, b, c\}$ and $S = \{b^2ab^{-1}a^{-1}, c^2bc^{-1}b^{-1}, a^2ca^{-1}c^{-1}, ab^{48}c^{-5151}\}$ then one writes

$$G \cong \langle a, b, c \mid b^2ab^{-1}a^{-1}, c^2bc^{-1}b^{-1}, a^2ca^{-1}c^{-1}, ab^{48}c^{-5151} \rangle.$$

Note however that a group may have very different presentations. For example, the above presentation in fact gives the trivial group. There is no algorithm that tells us, given finite sets A_1, A_2 and S_1, S_2 as above, whether the groups $\langle A_1 \mid S_1 \rangle$ and $\langle A_2 \mid S_2 \rangle$ are isomorphic. So the fundamental group is a powerful but often intractable invariant of topological spaces. It knows a lot but persuading it to share this information can be tricky.

One thing one can do is to abelianise the fundamental group. This normally results in a huge loss of information but nevertheless this trick suffices e.g. to tell apart surfaces without boundary. The *commutator subgroup* of a group G is the subgroup $G' \subset G$ generated by all commutators $[a, b] = aba^{-1}b^{-1}$ where $a, b \in G$.

Question 4. (a) Show that G' is indeed a subgroup of G and that it is normal. The quotient G/G' is called the *abelianisation* of G. Show that G/G' is *abelian*, i.e. that $gh = hg$ for all $g, h \in G/G'$. Note that every subgroup of an abelian group is normal.
(b) Suppose $G = \langle a_1, \ldots, a_n \mid w_1, \ldots, w_k \rangle$. For each w_i form an element $v_i = (v_i^1, \ldots, v_i^n) \in \mathbb{Z}^n$ as follows: we set v_i^j to be the sum of the degrees of all occurrences of a_j in w_i. E.g., if $n = 2$, then $w = a_1^{23}a_2^{-15}a_1^{48}a_2a_1^{10}$ would give $(81, -14)$, while $w = a_1^{100}a_2^{-100}a_1^{-100}a_2^{100}$ would give $(0, 0)$. Show that G/G' is isomorphic to the quotient

$$\mathbb{Z}^n/\langle v_1, \ldots, v_k \rangle.$$

The fundamental groups

Question 5. (a) Show that if a topological space X is path connected, then for every two elements $x_1, x_2 \in X$ we have an isomorphism $\pi_1(X, x_1) \cong \pi_1(X, x_2)$.

(b) Show that if two spaces X, Y are homotopy equivalent and path connected, then there is an isomorphism $\pi_1(X, x_0) \cong \pi_1(Y, y_0)$ for any choice of $x_0 \in X, y_0 \in Y$.

The fundamental groups of surfaces

In the lectures we have shown that if S is an orientable surface of genus g then

$$\pi_1(S) \cong \langle a_1, b_1, \ldots, a_g, b_g \mid [a_1, b_1] \cdot \cdots \cdot [a_g, b_g] \rangle$$

and if S is a non-orientable surface of genus g then

$$\pi_1(S) \cong \langle a_1, \ldots a_g \mid a_1^2 \cdot \cdots \cdot a_g^2 \rangle.$$

We will need another notion from group theory. If G, H are groups, then their cartesian product $G \times H$ is again a group under componentwise operation: $(g_1, h_1) \cdot (g_2, h_2) = (g_1 g_2, h_1 h_2)$. If both the groups are abelian it is customary to denote the resulting group as $G \oplus H$. The group $\mathbb{Z}^n$ of all n-tuples of integers under addition is nothing but $\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$ (n factors).

Question 6. (a) Show that $\mathbb{Z}^n \ncong \mathbb{Z}^m$ unless $n = m$. [A warning is perhaps in order: although we do not prove it here, $\mathbb{R}^n$ and $\mathbb{R}^m$ *are* isomorphic as groups (but, of course, not as real vector spaces).]

(b) Show that $\mathbb{Z}^n / \langle (2, 2, \ldots, 2) \rangle$ is isomorphic to $\mathbb{Z}/2 \oplus \mathbb{Z}^{n-1}$. Deduce that for all $m, n \geq 1$

$$\mathbb{Z}^n / \langle (2, 2, \ldots, 2) \rangle \ncong \mathbb{Z}^m$$

and that for $m \neq n, m, n \geq 1$

$$\mathbb{Z}^n / \langle (2, 2, \ldots, 2) \rangle \ncong \mathbb{Z}^m / \langle (2, 2, \ldots, 2) \rangle.$$

[Hint: show that if G is an abelian group then the elements of finite order form a subgroup; this subgroup is called the *torsion subgoup* of G and is denoted G_T; show that if abelian subgroups G_1 and G_2 are isomorphic then $G_{1T} \cong G_{2T}$ and $G_1/G_{1T} \cong G_2/G_{2T}$. Note however that for non-abelian groups the elements of finite order may not form a subgroup.]

(c) Using parts (a) and (b) and question 4 show that if surfaces S_1, S_2 without boundary are homotopy equivalent, then either both are orientable, or both are non-orientable, and in both cases the genera of the surfaces coincide.

So we have finally shown (albeit in a roundabout way) that for surfaces without boundary the genus and orientability are topological (and even homotopy) invariants.

Problems for discussion

Here we derive the **Wirtinger presentation** of the fundamental group of a link complement. First, recall that if G, H are groups the *free product* $G * H$ is the set of all seqences $x_1 * \cdots x_k$ where x_i belong to G or H, up to the equivalence relation generated by $s_1 * x * y * s_2 = s_1 * xy * s_2$ where s_1, s_2 are arbitrary sequences and x, y are both in G or both in H. Note that a free group on n generators is just the free product of n copies of $\mathbb{Z}$. Second, if K is a third group equipped with group homomorphisms $f : K \to G, g : K \to H$ then the *cofibre product* $G *_K H$ of G and H over K is the quotient of $G * H$ by the normal closure of the set $f(x) * g(x), x \in K$.

It can be shown that if $\langle A_1 \mid S_1 \rangle$ is a presentation for G and $\langle A_2 \mid S_2 \rangle$ is a presentation for H, then $\langle A_1 \cup A_2 \mid S_2 \cup S_2 \cup S \rangle$ is a presentation for $G *_K H$ where S is formed by all words $f(c_i) g(c_i)^{-1}$. Here c_i is an elenement of some system of generators of K and $f(c_i), g(c_i)$ are its images under f and g, written in terms of the generators from A_1, respectively A_2. For more details see A. Hatcher, Algebraic Topology, pp. 40-43.

Van Kampen's theorem says that if a path connected CW-complex X is the union of its path connected subcomplexes X_1, X_2 such that $X_2 \cap X_2$ is path connected, then for all $x_0 \in X_1 \cap X_2$ we have $\pi_1(X, x_0) \cong \pi_1(X_1, x_0) *_{\pi_1(X_1 \cap X_2, x_0)} \pi_1(X_2, x_0)$. This less general than the version in Hatcher's book, but for our purposes it'll do.

Let L be a link in $\mathbb{R}^3$. We equip $\mathbb{R}^3$ with coordinates x, y, z and then place the link so that most of it, apart from a finite number of short arcs, lies in the plane $z = 0$ (those arcs correspond to the overcrossings of the link diagram obtained by projecting to $z = 0$). We assume L to be oriented. Applying a planar isotopy we may deform the diagram of L so that all overcrossings are on the same line and all overcrossing strands look in the same direction.

By cutting everything using the plane $z = \varepsilon > 0$ represent $\mathbb{R}^3 \setminus L$ as a union $X_1 \cup X_2$ where X_1 is a half-space with some number of short tunnels drilled in it (those correspond to the overcrossing strands), X_2 is a half-space with some number of long tunnels drilled in it (those correspond to the rest of the link) and $X_1 \cap X_2$ is $\mathbb{R}^2$ with some number of non-intersecting holes, all on the same line l. Suppose the crossings are numbered from 1 to n. Choose a point $x_0 \in (X_1 \cap X_2) \setminus l$. For crossing number i define loops γ_i^+, γ_i^- in $X_1 \cap X_2$ based at x_0, as shown on the blackboard (if we move along a component of L in the positive direction then, as we cross over at i-th crossing the loops go counterclockwise).

1. Show that the classes of these loops freely generate $\pi_1(X_1 \cap X_2, x_0)$.

2. Show that X_1 is homotopy equivalent to a wedge of circles. Show that $[\gamma_i^+] = [\gamma_i^-]$ in $\pi_1(X_1, x_0)$ and that either of the sets $\{[\gamma_i^+]\}$ and $\{[\gamma_i^-]\}$ freely generates $\pi_1(X_1, x_0)$.

3. Show that X_2 is also homotopy equivalent to a wedge of circles and that its π_1 is freely generated by either of the sets $\{[\gamma_i^+]\}$ and $\{[\gamma_i^-]\}$.

4. Express each $[\gamma_i^-]$ (regarded as an element of $\pi_1(X_2, x_0)$) in terms of $[\gamma_i^+]$'s and derive a presentation of $\pi_1(\mathbb{R}^3 \setminus L, x_0)$.