HSE/Math in Moscow 2013-2014// Topology 1 // Problem sheet 12

Group actions

If G is a group and X is a set then a *left action* of G on X is a map $G \times X \to X, (g, x) \mapsto g \cdot x$ such that $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$ for all $g_1, g_2 \in G, x \in X$. Similarly, a *right action* of G on X is a map $X \times G \to X, (x, g) \mapsto x \cdot g$ such that $(x \cdot g_1) \cdot g_2 = x \cdot (g_1 g_2)$ for all $g_1, g_2 \in G, x \in X$. In the sequel we'll often be omitting the dots and we'll be simply writing gx instead of $g \cdot x$, when this is unlikely to cause confusion. If a (left or right) action of G on X is given, we say that X is a (left or right) G-set

Given a left action as above and an $x \in X$ we define the *orbit* $G(x)$ of x and the stabiliser G_x of x as follows $G(x) = \{gx \mid g \in G\}$, $G_x = \{g \in G \mid gx = x\}$. Orbit and stabilisers for right actions are defined in a similar way. Given a subgroup H of G let us also set $G/H = \{gH \mid g \in G\}$ and $H \setminus G = \{Hg \mid g \in G\}$; in other words, G/H is the set of all left cosets and $H \setminus G$ is the set of all right cosets of H. Note that we also use $\setminus$ to denote the difference of two sets but, again, hopefully this will never lead to a confusion, as in the latter meaning $H \setminus G$ is always empty.

Question 1. In this question and the following one G is a group and X is a left G-set.

(a) Given an $x \in X$, construct a bijection $G/G_x \cong G(x)$ and deduce the *orbit-stabiliser* theorem: if G is finite then $\#G(x)$ is always a divisor of $\#G$.

(b) Show that G acts on itself by conjugations and deduce that if $\#G$ is a power of a prime then G has a non-trivial centre (recall that the *centre* of a group G is the set of all $g \in G$ that commute with every $g' \in G$).

(c) Show that for all $g \in G, x \in X$ we have $G_{gx} = gG_xg^{-1}$.

A G-set X is *transitive* iff there is an $x \in X$ such that $G(x) = X$. If X, Y are G-sets and $f : X \to Y$ is a map then we say that f is a *map of G-sets* or a *G-map* if $f(gx) = gf(x)$ for all $g \in G, x \in X$. If f is bijective we say that X and Y are *isomorphic as G-sets* or *G-isomorphic*.

Question 2. (a) Show that if X is a transitive left G-set than $G(x) = X$ for all $x \in G$.

(b) Show that any transitive left G-set is isomorphic to G/H for some subgroup $H \subset G$.

(c) Now let H be a subgroup of G. Show that there exists a map $G/H \to G/H$ of left G-sets that takes $H = eH$ to gH iff $g \in N_G(H)$ where $N_G(H)$ is the *normaliser* of H in G, i.e., the largest subgroup of G in which H is normal.

(d) Deduce from the previous parts and part (c) of Question 1 that any G-map $G/H \to G/H$ is necessarily an isomorphism and the group of all G-isomorphisms $G/H \to G/H$ is isomorphic to $N_G(H)/H$.

Covering maps

A surjective continuous map $f : X \to Y$ is a *covering map* iff for all $y \in Y$ there is an open $U \ni y$ such that $f^{-1}(U)$ is homeomorphic to $U\times$ a discrete set; moreover, the homeomorphism is required to commute with the projections of both sets onto U.

Question 3. (a) For any two $g_1, g_2 \geq 1$ such that $(2 - 2g_2)$ divides $(2 - 2g_1)$ construct a covering $S_{g_1} \to S_{g_2}$ where S_g denotes an orientable compact connected genus g surface without boundary.

(b) Can a compact connected surface without boundary other than S^2 cover S^2?

Recall that an action of a discrete group G on a topological space X is *properly discontinuous* iff for all $x \in X$ there is an open $U \ni x$ such that $g(U) \cap U = \varnothing$ for all $g \in G, g \neq e$, the unit element of G. (Note that this definition has to be modified a bit if G itself has a non-trivial topology but we do not consider such groups here.)

Question 4. (a) Show that if G acts properly discontinuously on a path connected and locally path connected topological space X, then the map $p : X \to X/G$ is a covering map.

(b) Choose an $x_0 \in X/G$ and a loop γ in X/G based at x_0. Show that either all lifts of γ to X are loops, or all of them are non-loops.

(c) In the lectures we have constructed an action of $\pi_1(X/G, x_0)$ on $p^{-1}(x_0)$ such that the stabiliser of $\tilde{x}_0 \in p^{-1}(x_0)$ is precisely $p_*(\pi_1(X, \tilde{x}_0))$. Use this and part (c) of Question 1 to show that the latter group is normal in $\pi_1(X/G, x_0)$.

Miscellany

Question 5. Show that if a topological space X is represented as a union $Y \cup Z$ of its closed subspaces and $f : Y \to W$ is a continuous map then $X \cup_f W$ is homeomorphic to $Y \cup_{f|_{Y \cap Z}} W$.

Question 6. Set $X = D^n$, $U = U^n \subset X$ and let $f : X \to Y$ be a map from X to a Hausdorff space Y that is injective on U. Show that the following statements are equivalent:

(a) $f|U$ is a homeomorphism onto its image.

(b) $f(U) \cap f(X \setminus U) = \varnothing$.

[Hint: to show (a)$\Rightarrow$(b) join a point $x \in S^{n-1}$ such that $f(x) = f(u), u \in U$ with point in U by a segment that does not contain u; then try to find a closed subset of U whose image is not closed in $f(U)$; to show (b)$\Rightarrow$(a) show that if (b) then for a closed $Z \subset U$ we have $f(\bar{Z}) \cap f(U) = f(Z)$.]

Problems for discussion

In this last discussion session we consider the Riemann-Hurwitz formula and some of its applications.

Here is a slight variation on the notion of a covering map: suppose M, N are surfaces and let $f : M \to N$ be a continuous map. We say that f is a *branched covering* if it is a genuine covering map over N minus a discrete set. The minimal $K \subset N$ such that $f|_{f^{-1}(N\setminus K)} : f^{-1}(N \setminus K) \to N \setminus K$ is a covering map is called the *branch locus* or *ramification locus* of f.

It can be shown that if f is a branched covering then for any $x \in M$ there are open sets $U \ni x, V \ni f(x)$ and homeomorphisms $\varphi : U \to U' \subset \mathbb{C}, \psi : V \to V' \subset \mathbb{C}$ such that U', V' are open in $\mathbb{C}$, contain the origin and $\psi \circ f \circ \varphi^{-1}$ is given by $z \mapsto z^n, n > 0$. In other words f can locally be written as $z \mapsto z^n$. The integer n is called the *local multiplicity* of f at x; we'll denote it μ_x. It can be shown that the local multiplicity is well-defined, i.e., that it does not depend on U, V, φ, ψ. The set of all $x \in M$ such that the local multiplicity of f at x is > 0 is called the *singular locus* of f and is denoted $Sing(f)$. The image of the singular locus is contained in the ramification locus.

Branch covers are far more abundant than covering maps. For instance, a *Riemann surface* is a surface M such that for every $x \in M$ there is an open $U_x \ni x$ and a homeomorphism $\varphi_x : U_x \to U'_x$ (where U'_x is an open subset of $\mathbb{C}$) such that if $U_x \cap U_y \neq \varnothing$ then $\varphi_x \circ \varphi_y^{-1}$ is (on its domain of definition $\subset \mathbb{C}$) a homolorphic map with nowhere vanishing derivative. This allows one to define *holomorphic* maps between Riemann surfaces; the definition generalises that of a holomorphic map between open subsets of $\mathbb{C}$. It turns out that every holomorphic map from one compact Riemann surface to another is a branched cover (note that genuine covering maps are relatively rare).

In the sequel $f : M \to N$ is a continuous map of compact surfaces without boundary.

1. Show that if f is a genuine covering map then for all $y \in N$ the preimage $f^{-1}(y)$ contains finitely many elements and show that $\chi(M) = n\chi M$ where $n = \#f^{-1}(y), y \in N$.

2. From now on we suppose that f is a branched covering and let K be the ramification locus. Show that for all $y \in N$ the preimage $f^{-1}(y)$ contains finitely many elements

$$\chi(M) = (\chi(N) - \#K)n + n$$

where $n = \#f^{-1}(y), y \in N \setminus K$. This is the first version of the Riemann-Hurwitz formula.

3. Show that for any $y \in N$ (which may or may not belong to K) we have $\sum_{x\in f^{-1}(y)} \mu_x = n$. One way to rephrase this would be that the preimages of all points of N contain the same number of elements, provided one counts them with multiplicities. Deduce that

$$\chi(M) = n\chi(N) - \sum_{x\in Sing(f)} (\mu_x - 1).$$

This is the second version of the Riemann-Hurwitz formula.

4. Now let M be the set of all $(x : y : z) \in \mathbb{C}P^2$ such that $x^k + y^k = z^k$ and set $N = \{(0 : y : z) \in \mathbb{C}P^2\} \cong \mathbb{C}P^1$. Set $Z = \mathbb{C}P^2 \setminus \{(1 : 0 : 0)\}$ and define a map $f : Z \to N$ as follows: for each $P \in Z$ trace a projective line through P and $(1 : 0 : 0)$ and set $f(P)$ to be the point where this line intersects Z.

(a) Show that $M \subset Z$ and that f is well defined.

(b) Using the first version of the Riemann-Hurwitz formula show that, assuming M is a compact surface without boundary, $\chi(M) = n(3 - n)$. Assuming M is orientable, deduce that the genus of M is $\frac{(k-1)(k-2)}{2}$.

Remark. From this one can deduce that the genus of every smooth degree k curve in $\mathbb{C}P^2$ is $\frac{(k-1)(k-2)}{2}$.

5. Suppose M is a surface without boundary and that a finite group G acts on M so that the quotient $N = M/G$ is again a surface and the natural map $p : M \to N$ is a branched covering. One can show that this is always the case when M is a Riemann surface and M acts by biholomorphic transformations (e.g., holomorphic homeomorphisms whose inverses are again holomorphic) but we will not attempt this. Furthermore, we assume the action to be *faithful* (only the unit element of G acts as the identity transformation) and we set $n = \#G$.

(a) Show that for all $y \in N$ we have $\#f^{-1}(y)|n$.

From now on we suppose $\chi(M) < 0$. Set $k = \#K$. We want to find an upper bound on n in terms of $\chi(M)$.

(b) Show that if $k = 0$ then $n \leq -\chi M = |\chi M|$.

(c) From now on we suppose that, in addition to the above, $k > 0$. Using the Riemann-Hurwitz formula show that if $\chi N \leq 0$ then $\chi(M) \leq -\frac{kn}{2}$ and deduce that $n \leq -2\chi(M) = 2|\chi M|$.

From now on we suppose that, in addition to all the above, $\chi(N) > 0$, which makes N a real projective plane or S^2.

(d) Show that if $N \cong \mathbb{R}P^2$ then $\chi(M) \leq -\frac{kn}{2} + n$. Deduce that $k \leq 3$, then $n \leq -2\chi(M) = 2|\chi M|$ as above. Using Riemann-Hurwitz again show that $k = 1$ is impossible and if $k = 2$ then $-\chi(M) \geq \frac{n}{6}$, which gives $n \leq -6\chi(M) = 6|\chi M|$. [Hint: for any integers $a, b > 1$ the sum $\frac{1}{a} + \frac{1}{b}$ is either 1 or $\leq \frac{5}{6}$.]

(e) In a similar way show that if $N = S^2$, then $n \leq -42\chi(M) = 42|\chi M|$. [Hint: as in part (d), one shows that $\chi(M) \leq -\frac{kn}{2} + 2n$, and then gets the cases $k \geq 5$ and $k = 1, 2$ out of the way. Then, to see what happens when e.g. $= k3$, one might try to show first that if a, b, c are integers > 1, then $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$ is either ≥ 1 or $\leq$ some rational number < 1; which number? and why? The case $k = 4$ is similar.]

So in each of the above cases $n \leq 42|\chi M|$. Notice that this depends only on M, not on N. In other words, using a topological argument we have shown that groups which are too large cannot act in a nice way on a given surface, provided the Euler characteristic is negative. When the surface in question is a Riemann surface, this becomes the **Hurwitz bound** on the order of the automorphism group of a smooth complex projective curve.