HSE/Math in Moscow 2013-2014// Topology 1 // Problem sheet 5

Miscellany

Question 1. Show that a closed subspace of a complete metric space is again complete.

Question 2. Show that if X is a metric space and $Y \subset X$ is a subspace then $x \in \bar{Y}$ iff it is the limit of a sequence (y_n) of elements of Y.

Let $(x_n)_{n\in\mathbb{Z}_{\geq 0}}$ be a sequence of elements of a metric space (X, d) and let $j \mapsto n_j$ be an increasing map $\mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$. Then the sequence $(y_j), y_j = x_{n_j}$ is called a *subsequence* of (x_n).

Question 3. (a) Show that if $x = \lim_{n\to\infty} x_n$ then x is also the limit of any subsequence of (x_n).

(b) Show that x is not the limit of any subsequence of (x_n) iff there is an open set $U \ni x$ and $n \in \mathbb{Z}_{\geq 0}$ such that $U \not\ni x_m$ for all $m > n$.

Compactness criteria for metric spaces

For a metric space X the following statements are equivalent.

1. X is compact.
2. Every sequence of elements of X has a converging subsequence.
3. X is complete and *totally bounded* (i.e., for all $\varepsilon > 0$ there is a finite subset $K(\varepsilon) \subset X$ such that every $x \in X$ is $< \varepsilon$ away from an element of $K(\varepsilon)$).
4. Every infinite subset of $Y \subset X$ has a *limit point* (i.e., there is an $x \in X$ such that any neighbourhood of it contains infinitely many elements of Y).
5. Every sequence $Y_1 \supset Y_2 \supset \cdots \supset Y_n \supset \cdots$ of non-empty nested closed subsets of X has non-empty intersection.
6. X is *countably compact*, i.e. every countable open cover of X has a finite subcover.

A strategy for proving this would be to show that $1 \Rightarrow 2, 2 \Rightarrow 3, \ldots, 6 \Rightarrow 1$. We have partially realised this plan in the lectures.

Question 4. Prove the remaining implications.

For general spaces these properties are not equivalent (that is, those of them that even make sense without a metric).

Problems for discussion

Let $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ be a bounded continuous function and suppose there is a $C \in \mathbb{R}$ such that

$$||f(x_1, t) - f(x_2, t)|| \leq C||x_1 - x_2|| \tag{1}$$

for all $(x_1, t), (x_2, t) \in \mathbb{R}^n \times \mathbb{R}$ (where $|| \cdot ||$ denotes the standard Euclidean norm on $\mathbb{R}^n$). We want to prove the existence and uniqueness theorem for ODE's, i.e., that there is an $\varepsilon > 0$ such that for any $x_0 \in \mathbb{R}^n$ there is a solution $x : (-\varepsilon, \varepsilon) \to \mathbb{R}^n, t \mapsto x(t)$ of the differential equation

$$x'(t) = f(x(t), t) \tag{2}$$

that satisfies $x(0) = x_0$. Moreover, this solution is unique, depends continuously on x_0 and extends to a global solution of (2).

1. Show that (1) is satisfied provided f is differentiable everywhere and $||(f^i_{x_1}, \ldots, f^i_{x_n})||$ is bounded for each component f^i of $f = (f^1, \ldots, f^n)$.

2. Show that (2) is equivalent to

$$x(t) = x_0 + \int_0^t f(x(s), s)ds. \tag{3}$$

3. Show that there is an $\varepsilon > 0$ such that

$$x(t) \mapsto x_0 + \int_0^t f(x(s), s)ds$$

defines a contracting map $C^b([-\varepsilon,\varepsilon],\mathbb{R}^n) \to C^b([-\varepsilon,\varepsilon],\mathbb{R}^n)$. Deduce that the solution of (2) that satisfies $x(0) = x_0$ exists and is unique.

4. Prove the following generalisation of the contracting mapping theorem: Let (X, d) be a complete metric space. Suppose T is a topological space and $f : X \times T \to X$ is a map such that for all $x \in X$ the map $T \ni t \mapsto f(t, x) \in X$ is continuous. For $t \in T$ set $f_t(x) = f(x, t)$. Assuming each f_t is a contracting mapping with coefficient $q_t \in (0, r), r < 1$, show that the map

$$t \mapsto \text{the fixed point of } f_t$$

is continuous. [Hint: let x_t be the fixed point of f_t; show that for all $x \in X, t \in T$ we have $d(x, x_t) \leq \frac{1}{1-q_t} d(x, f_t(x))$; deduce that $d(x_{t'}, x_t) \leq \frac{1}{1-r} d(x_{t'}, f_t(x_t')) = \frac{1}{1-r} d(f_{t'}(x_{t'}), f_t(x_t'))$ for all $t, t' \in T$ and use the continuity of $t \mapsto f(t, x)$.]

Deduce that the solution from the previous question depends continuously on x_0.

5. Show that the solution from part 3 exists globally, i.e. extends to a map $\mathbb{R} \to \mathbb{R}^n$ that solves (2).

Remarks. In practical applications it often happens that f is defined on an open subset of $\mathbb{R}^n \times \mathbb{R}$ rather than on the whole space. It can also happen that the differential of some component of f is not bounded, even if f is everywhere defined. In either of these cases one can deduce similar theorems by replacing f with anther function $\bar{f}$ that coincides with f in a neighbourhood U of some point and is constant outside some $V \supset U$. One way to obtain such an $\bar{f}$ would be to multiply f by some function $g : \mathbb{R}^{n+1} \to \mathbb{R}$ that equals 1 in a small ball B and 0 outside a slightly larger ball with the same centre. Such a g can in turn be obtained from the function $h : \mathbb{R} \to \mathbb{R}$ such that $h(x) = 0$ for $x \in (-\infty, 0]$ and $h(x) = e^{-\frac{1}{x}}$ for $x \in (0, \infty)$. An easy check shows that h is infinitely differentiable.

So using question 4 above one can choose an $\varepsilon > 0$ and a ball $B' \subset \mathbb{R}^n$ so that the solutions $x(t)$ of the equation $x' = \bar{f}(x, t)$ such that $x(0) \in B'$ have the property that $(t, x(t)) \in B$ for $t \in (-\varepsilon, \varepsilon)$, and so solve the original equation (2). However, this time there is no guarantee that these solutions exist for all t: it can happen that some or all solutions blow up (escape to infinity in finite time). A simple equation for which this happens is $x' = -x^2$: its solutions are $x(t) = 0$ and $x(t) = \frac{1}{t-C}, C \in \mathbb{R}$.