HSE/Math in Moscow 2013-2014// Topology 1 // Problem sheet 6

Real projective spaces

In this part we'll often have to consider quotient spaces, so let us first recall a useful notion from set theory. Suppose X is a set and $\{(x_\alpha, y_\alpha)\}_{\alpha \in A}$ is a subset of X^2. Then the *equivalence relation* $\sim$ *generated by* $x_\alpha \sim y_\alpha, \alpha \in A$ is defined as follows: $x \sim y$ iff $x = y$ or there is a finite sequence

$$x = x_0, x_1, \ldots, x_n = y$$

such that for any i there is an $\alpha \in A$ for which $x_i = x_\alpha, x_{i+1} = y_\alpha$ or $x_i = y_\alpha, x_{i+1} = x_\alpha$. This is the finest equivalence relation under which each x_α becomes equivalent to y_α. We will usually denote the quotient as $X/x_\alpha \sim y_\alpha, \alpha \in A$. For example, $[0,1]/0 \sim 1$ is the quotient of $[0,1]$ with respect to the following equivalence relation: $x \sim y$ iff $x = y$ or $x = 0, y = 1$ or $x = 1, y = 0$.

If X is a topological space equipped with an equivalence relation $\sim_X$ and $Y \subset X$ is a subspace, then $\sim_X$ induces an equivalence relation $\sim_Y$ on Y. We have a natural map $Y/\sim_Y \to X/\sim_X$.

Question 1. (a) Show that this map is continuous.

(b) Set $X = [0,1]$ and set Y to be the subset whose elements are 0, $\frac{1}{n}, n \in \mathbb{Z}_{>1}$ and $1 - \frac{1}{n}, n \in \mathbb{Z}_{>1}$. Let $\sim$ be the equivalence relation generated by $0 \sim 1$. Show that the map $Y/\sim_Y \to X/\sim_X$ is not a homeomorphism onto its image.

Nevertheless, if Y is compact and $X/\sim_X$ is Hausdorff, these kind of problems do not occur, which might come in useful in the next question.

Recall that the real projective space $\mathbb{R}P^n$ is $\mathbb{R}^{n+1} \setminus \{0\}/\sim$ where $x \sim y$ iff there is a $t \in \mathbb{R} \setminus \{0\}$ such that $tx = y$. The equivalence class of $(x_0, \ldots, x_n)$ is denoted $(x_0 : \ldots : x_n)$.

Set

$$S^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \sum x_i^2 = 1\}, D^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid \sum x_i^2 \leq 1\}.$$

Question 2. Show that $\mathbb{R}P^n$ is homeomorphic to

(a) $S^n/\sim$ where $x \sim y$ iff $x = \pm y$.

(b) $D^n/\sim$ where $x \sim y$ iff $x = y$ or both $x, y \in S^{n-1}$ and $x = \pm y$.

Surfaces

Question 3. Show that $\mathbb{R}P^2$ minus a small open disk is homeomorphic to the Möbius strip.

Question 4. (a) Let $\sim$ be the equivalence relation on the cylinder $X = S_1 \times [0,1]$ generated by $(x,0) \sim (-x,0)$. Show that $X/\sim$ is homeomorphic to the 2-torus.

(b) Now let $\sim$ be the equivalence relation on the cylinder X generated by $(x,0) \sim (\bar{x},0)$ where we view S^1 as the unit circle in $\mathbb{C}$ and so $\bar{x}$ is the complex conjugate of x. Show that $X/\sim$ is homeomorphic to the Klein bottle.

Question 5. Show that $T^2 \# \mathbb{R}P^2$ is homeomorphic to $\mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2$.

Homotopy and homotopy equivalence

Let X, Y be topological spaces.

Recall that two continuous maps $f_0, f_1 : X \to Y$ are *homotopic* iff there is a continuous map $F : X \times [0,1] \to Y$ such that for all $x \in X$ we have $f_0(x) = F(x,0), f_1(x) = F(x,1)$.

Question 6. Show that being homotopic is an equivalence relation on the set of all continuous mapr $X \to Y$. [Hint: use question 2 from problem sheet 1.]

We say that X, Y are *homotopy equivalent* iff there are continuous maps $f : X \to Y, g : Y \to X$ such that $f \circ g \sim id_Y, g \circ f \sim id_X$.

Question 7. (a) Let X, Y, Z be topological spaces. Show that if continuous maps $f_0, f_1 : X \to Y$ and $g_0, g_1 : Y \to Z$ are homotopic, then so are $g_0 \circ f_0$ and $g_1 \circ f_1$. Deduce that if X is homotopy equivalent to Y and Y is homotopy equivalent to Y, then X is homotopy equivalent to Z.

(b) Let $X \subset \mathbb{R}^n$ be a subset with the property that there is an $x_0 \in X$ such that for all $x \in X$ the segment that joins x and x_0 is included in X. For example, all convex sets have this property: one can take any point as x_0. Show that X is homotopy equivalent to $\{x_0\}$.

(c) Show that $\mathbb{R}^n \setminus \{0\}$ is homotopy equivalent to S^{n-1}.

Problems for discussion

Recall that a *half space* of $\mathbb{R}^n$ is the set of all $x \in \mathbb{R}^n$ such that $f(x) \geq 0$ where $f(x_1, \dots, x_n) = \sum a_i x_i + a, a_i, a \in \mathbb{R}$. A *polytope* or a *polyhedron* is the intersection of finitely many half-spaces. A polytope in $\mathbb{R}^n$ is *regular* iff its isometry group acts transitively on the set of all collections $(f_0, \dots, f_{n-1})$ such that $f_0 \subset \dots \subset f_{n-1}$, where f_i is a face of dimension i.

We want to show that a regular polytope P in $\mathbb{R}^3$ has the same number of vertices, edges and 2-faces (denote these numbers v, e, f respectively) as a polytope from the following list: a simplex, a cube, an octahedron, a dodecahedron, an icosahedron.

1. Prove that all 2-faces of P have the same number of edges and at all vertices the same number of edges meet. Denote these numbers k and n respectively. Explain why $v - e + f = 2$ using Euler's formula from problem sheet 1 or the Euler characteristic.

2. Show that $e = \frac{fk}{2}, v = \frac{fk}{n}$.

We will assume that $n, k \geq 3$.

3. Show that $k < 6$.

4. Show that P has the same number of vertices, edges and 2-faces as

a) a simplex, an octahedron or an icosahedron if $k = 3$;

b) a cube if $k = 4$;

c) a dodecahedron if $k = 5$.

Using Euclidean geometry one can show that all the regular polytopes from the above list do in fact exist and are unique up to an isometry of $\mathbb{R}^3$, possibly composed with a homothety (dilation). Topology cannot tell us that. Note however that since our argument uses no geometry at all, apart from the assumption $n, k \geq 3$, our conclusion is also valid for polytopes with spherical boundary in spherical or hyperbolic geometry or, more generally, in any Riemannian 3-manifold for which the definition of a regular polytope makes sense.