HSE/Math in Moscow 2013-2014// Topology 1 // Problem sheet 2

Compact spaces, Hausdorff spaces and homeomorphisms

A very important notion in topology is that of a compact space. An *open cover* of a topological space X is a collection $\{U_i\}_{i\in I}$ of open subsets such that $\bigcup_{i\in I} U_i = X$. We say that X is *compact* iff any open cover $\{U_i\}_{i\in I}$ of X contains a finite subcover, i.e., there is a *finite* subset $J \subset I$ such that $\bigcup_{j\in J} U_j = X$. For example, every finite topological space is compact. We will assume that the closed unit interval $[0, 1]$ is compact. A typical non-example is $\mathbb{R}$: it can be covered by the open intervals $(-n, n), n \in \mathbb{Z}_{>0}$.

Question 1. a) Let $f : X \to Y$ be a continuous map of topological spaces. Prove that if $A \subset X$ is compact (in the topology induced by the inclusion in X) then so is $f(A)$ (in the topology induced by the inclusion in Y).

b) Give an example of a continuous map $f : X \to Y$ and a compact subset B of Y such that $f^{-1}(B)$ is not compact.

A topological space X is *Hausdorff* iff for any $x, y \in X$ such that $x \neq y$ there are open subsets $U \ni x$ and $V \ni y$ such that $U \cap V = \varnothing$. A typical example would be the Euclidean space $\mathbb{R}^n$ or any of its subspaces.

Remark. In some sources compact spaces are required, in addition to the above property, to be Hausdorff, while spaces which only have the above property but are not necessarily Hausdorff are called quasi-compact. In this course we will encounter very few non-Hausdorff spaces, if any. Nevertheless, every now and again we will have to check that something that we have constructed is Hausdorff.

Question 2. Show that if X is Hausdorff and A is compact then A is closed in X.

Question 3. Show that a closed subset of a compact space is again compact.

Question 4. a) Show that if X is compact and Y is Hausdorff, then any continuous bijection $f : X \to Y$ is a homeomorphism. [Hint: show that f takes closed sets to closed sets.]

b) In problem sheet 1 we saw that the compactness of X is necessary in order for the conclusion from part a) to be true. Show that the condition that Y be Hausdorff is necessary as well.

The Cartesian product $X \times Y$ of topological spaces X and Y can be equipped with *the product topology*, in which $W \subset X \times Y$ is open iff for every $(x, y) \in W$ there are open subsets $U \ni x$ of X and $V \ni y$ of Y such that $U \times V \subset W$. Note that this generalises the topology we introduced on $\mathbb{R}^n$ in the lectures.

As a side remark, the product of an infinite family of topological spaces can be topologised in a similar way, and it turns out that if all the factors are compact, so is their product. This fact is called Tikhonov's theorem and it has several important applications, but we will neither need it nor prove it in this course.

Recall that a subset X of $\mathbb{R}^n$ is *bounded* iff X is contained in some (closed or open) ball.

Question 5. Show that $[0, 1]^n$ is compact. Deduce that a subset of $\mathbb{R}^n$ is compact iff it is closed and bounded.

Quotient topology

Question 6. Set X to be the *Stiefel manifold*

$$V_m(\mathbb{R}^n) = \{(v_1, \ldots, v_m) \in \mathbb{R}^n \times \cdots \times \mathbb{R}^n \mid v_1, \ldots, v_m \text{ are linearly independent}\}.$$

We set $(v_1, \ldots, v_m) \sim (v'_1, \ldots, v'_m)$ iff $v_1, \ldots, v_m$ and $v'_1, \ldots, v'_m$ span the same m-plane. Show that $X/\sim$ is compact, Hausdorff, and each of its points has a neighbourhood homeomorphic to $\mathbb{R}^{m(n-m)}$. This space will be called the *Grassmannian of m-planes in* $\mathbb{R}^n$ and denoted $G_m(\mathbb{R}^n)$. [Hint: if $\mathbb{R}^n = V \oplus W$ then one can view V as the graph of a linear mapping $V \to W$, namely, the zero mapping. What other vector subspaces of $\mathbb{R}^n$ of dimension $\dim V$ can be regarded as graphs of linear mappings $V \to W$?]

If $m = 1$, the resulting space is called the $n-1$ dimensional *real projective space* and is denoted $\mathbb{R}P^{n-1}$ or $\mathbb{P}^{n-1}(\mathbb{R})$.

In a similar way one defines the complex projective space $\mathbb{C}P^n$ and the quaternionic projective space $\mathbb{H}P^n$, and also complex and quaternionic Grassmannians, denoted $G_m(\mathbb{C}^n)$ and $G_m(\mathbb{H}^n)$ respectively. In the quaternionic case there are two options: one can consider either left or right quaternionic subspaces. This does not make much difference, as every left quaternionic subspace can be turned into a right one, and vice versa, via the quaternionic conjugation $a + bi + cj + dk \mapsto a - bi - cj - dk, a, b, c, d \in \mathbb{R}$, so the resulting quaternionic Grassmannians are homeomorphic.

Suppose we have two topological spaces X, Y and a continuous map $f : X' \to Y$ where X' is a subspace of X. Then we can define an equivalence relation $\sim$ on $X \sqcup Y$ as follows: for any $x_1, x_2 \in X'$ such that $f(x_1) = f(x_2) = y \in Y$ we set $x_1 \sim x_2 \sim y$. The quotient space will be denoted as $X \cup_f Y$ and will be called the *result of attaching X to Y along f*. In the particular case when Y is a point we will write X/X' instead of $X \cup_f Y$.

Ths *mapping cylinder* $Cyl(f)$ of a continous map $f : X \to Y$ is defined as $(X \times [0, 1]) \cup_{f_0} Y$ where $f_0 : X \times \{0\} \to Y$ is given by $f_0(x, 0) = f(x)$. The space $Cone(f) = Cyl(f)/(X \times \{1\})$ will be called the *mapping cone* of f.

Question 7. Show that the mapping cone of the natural map $S^n \to \mathbb{R}P^n$ is homeomorphic to $\mathbb{R}P^{n+1}$.

Problems for discussion

In this part we consider S^1 as the unit circle in $\mathbb{C}$.

1. We define a map $E : \mathbb{R} \to S^1$ by setting $E(x) = e^{2\pi i x}$. Show that if $f : S^1 \to S^1$ is continuous, then it can be lifted to a continuous map $\bar{f} : \mathbb{R} \to \mathbb{R}$, i.e., there is a continuous map $\bar{f} : \mathbb{R} \to \mathbb{R}$ such that $f \circ E = E \circ \bar{f}$. Show that if $F : S^1 \times [0,1] \to S^1$ is a homotopy then there is a homotopy $\bar{F} : \mathbb{R} \times [0,1] \to \mathbb{R}$ that lifts F, i.e., $F \circ E = E \circ \bar{F}$. (All this is a special case of the homotopy lifting property, which we will prove later on in this course.)

2. Show that if $\bar{f}_1, \bar{f}_2 : \mathbb{R} \to \mathbb{R}$ lift the same $f : S^1 \to S^1$, then there is an $n \in \mathbb{Z}$ such that $\bar{f}_1(t) - \bar{f}_2(t) = n$ for all $t \in \mathbb{R}$. Show that there is a $d \in \mathbb{Z}$ such that if $\bar{f} : \mathbb{R} \to \mathbb{R}$ is any lift of f, then $\bar{f}(t+1) - \bar{f}(t) = d$. This integer d is called the *degree* of f.

3. Show that homotopic maps $S^1 \to S^1$ have the same degrees. Conversely, show that if two maps $S^1 \to S^1$ have the same degrees, they are homotopic.

4. Calculate the degree of the map $S^1 \ni z \mapsto z^n \in S^1$.

5. Let $f(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a complex polynomial. Suppose f has no complex roots. For each $t \in \mathbb{R}_{\geq 0}$ define a map $f_t : S^1 \to S^1$ by setting $f_t(z) = \frac{f(tz)}{|f(tz)|}$. Show that if $t \geq 2n \max(a_i)$ then f_t is homotopic to $z \mapsto z^n$. [Hint: consider the family $s \mapsto z^n + s(a_{n-1}z^{n-1} + \cdots + a_1 z + a_0)$ of polynomials.] Deduce the "main theorem of algebra".

6. Let D^2 denote the unit disk $\subset \mathbb{C}$. Let $f : D^2 \to D^2$ be a continuous map. Suppose f has no fixed points. Define $u_t : S^1 \to S^1, t \in [0,1]$ by $u_t(z) = g(tz)$ where $g : D^2 \to S^1$ is given by $g(z) = \frac{f(z)-z}{|f(z)-z|}$. Show that the degree of u_0 is 0. Show that u_1 is homotopic to the map $z \mapsto -z$. Calculate the degree of the latter map and deduce Brouwer's theorem in dimension 2.