HSE/Math in Moscow 2013-2014// Topology 1 // Problem sheet 4

Bases

A family $\mathcal{U}$ of open subsets of a topological space X is a called a *base* iff for any $x \in X$ and any open $V \ni x$ there is a $U \in \mathcal{U}$ such that $x \in U \subset V$. Many interesting spaces turn out to have a countable base of open sets. For example, open intervals $(a, b), a, b \in \mathbb{Q}$ form a base for the usual topology of $\mathbb{R}$.

A topological space is called *separable* iff it has a countable dense subset. For example, $\mathbb{R}$ is separable as $\mathbb{Q} \subset \mathbb{R}$ is dense and countable.

Question 1. (a) Show that $\mathbb{R}^n$ with the usual topology is separable and has a countable base of open subsets.

(b) Show that if a topological space X has a countable base $\mathcal{U}$ and $Y \subset X$ is a subspace, then $U \cap Y, U \in \mathcal{U}$ form a countable base for Y.

Question 2. Show that a family $\mathcal{U}$ of open subsets of a topological space X is a base iff every open $U \subset X$ can be represented as the union of some collection of subsets from $\mathcal{U}$.

Question 3. (a) Show that if a topological space has a countable base, it is separable.

(b) Show that if a *metric* space is separable, it has a countable base.

So a metric space is separable iff it has a countable base, and in this case each of its subsets is again separable. Neither this conclusion, nor the conclusion of 1 (b) is true for general spaces but we'll not dwell on this.

Hausdorff metric

If (X, d) is a metric space, $x \in X$ and $Y \subset X$, then we set $d(x, Y) = \inf_{y \in Y} d(x, y)$.

Question 4. (a) Show that if Y is compact then we could replace inf by max in this definition, i.e., the infimum is attained at some $y \in Y$.

(b) Show by example that in general, even if Y is closed in X, nevertheless it can happen that the infimum is not attained. [Hint: take X equal $\mathbb{R}^2$ minus a point.]

Given a metric space (X, d) and two non-empty subsets $Y, Z \subset X$ we define the *Hausdorff distance* $d_H(Y, Z)$ between Y and Z as

$$d_H(Y, Z) = \max(\sup_{y \in Y} d(y, Z), \sup_{z \in Z} d(z, Y)) \in [0, +\infty].$$

Question 5. Show that for all $Y, Z, W \subset X$ non-empty the triangle inequality holds: $d_H(Y, W) \leq d_H(Y, Z) + d_H(Z, W)$. [Hint: set $d_{YZ} = d_H(Y, Z), d_{Z,W} = d_H(Z, W)$; take a $y \in Y$ and show that for all $\varepsilon > 0$ there is a $w \in W$ such that $d(y, w) < d_{YZ} + d_{ZW} + \varepsilon$; deduce that $d(y, W) \leq d_{YZ} + d_{ZW}$.]

Question 6. (a) Show that if $Y, Z \subset X$ are non-empty, then $d(Y, Z) = 0$ iff $\bar{Y} = \bar{Z}$.

(b) Give an example of two non-empty closed subsets of a metric space such that the Hausdorff distance between them is infinite.

So the Hausdorff distance is not really a metric in the sense of problem sheet 3. Nevertheless, if we fix a non-empty closed $Y \subset X$ we do get a genuine metric on the set of all non-empty closed $Z \subset X$ such that $d_H(Y, Z) < \infty$. If X has finite diameter (i.e., if there is a $D \in \mathbb{R}$ such that $d(x, y) \leq D$ for all $x, y \in X$), we get a metric on the set of all closed subsets.

Continuity vs uniform continuity

Let $f : (X, d_X) \to (Y, d_Y)$ be a map of metric spaces. We say that f is continuous iff for all $x \in X$ and $\varepsilon > 0$ there is a $\delta = \delta(x, \varepsilon)$ (which depends on x and ε) such that $d_X(x, x') < \delta$ implies that $d_Y(f(x), f(x')) < \varepsilon$. We say that f is *uniformly continuous* iff for all $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon)$ (which depends only on ε) such that $d_X(x, x') < \delta$ implies that $d_Y(f(x), f(x')) < \varepsilon$. In the lectures we have seen Weierstrass's theorem; it says that if X is compact then any continuous $f : (X, d_X) \to (Y, d_Y)$ is uniformly continuous.

Question 7. (a) Show that the function $f : \mathbb{R} \to \mathbb{R}, f(x) = x^2$ is not uniformly continuous if $\mathbb{R}$ is equipped with the standard metric.

(b) Introduce another metric on $\mathbb{R}$ that induces the standard topology and such that f from part (a) becomes uniformly continuous. There are two $\mathbb{R}$'s here, the source and the target. You are allowed to change the metric on either of them, or both. [Hint: try to identify $\mathbb{R}$ with $(0, 1)$ so that the map extends to the endpoints $0, 1$ and use Weierstrass's theorem.]

So uniform continuity, as opposed to just continuity, depends on the metric, not just on the topology.

Problems for discussion

Let τ be the space drawn on the blackboard. The point where the three segments meet will be called the *centre* of τ. Let $X \subset \mathbb{R}^2$ be a disjoint union of homeomorphic copies of τ. We want to show that there are at most countably many such copies. Clearly, one can embed uncountably many disjoint τ's into $\mathbb{R}^3$, so if we succeed, we'll in particular prove that $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^3$.

1. Prove that in order to show that it suffices to show that there are at most countably many τ's all three of whose tails intersect a given circle and whose centre is inside the circle. [Hint: $\mathbb{R}^2$ is separable (and Hausdorff); the Intermediate Value Theorem might also come in useful here.]

Now let C be a circle in $\mathbb{R}^2$. We shall say that 3-element subsets $K_1, K_2 \subset C$ are *separated* from one another iff they lie in disjoint arcs of C (see blackboard).

2. Given a τ' whose centre is inside C construct a 3-element subset $K(\tau') \subset C$ such that if τ', τ'' are two copies of τ with centres inside C then $K(\tau')$ and $K(\tau'')$ are separated.

3. Now equip $\mathbb{R}^2$ with the standard metric. Let $B(C, 3)$ be the set of all 3-element subsets of C equipped with the induced Hausdorff metric. Show that $B(C, 3)$ is separable and deduce that it can't contain uncountably many disjoint open balls.

4. Take a $K \in B(C, 3)$. Show that there is an ε depending on K such that no $K' \in U(K, \varepsilon)$ is separated from K. Deduce that the number of copies of τ with centres inside C and all the three tails intersecting C is at most countable.