# HSE/Math in Moscow 2013-2014// Topology 1 // Problem sheet 3

**Metric spaces**

Recall that a *metric space* is a couple $(X, d)$ where $X$ is a set and $d$ is a function $X \times X \to \mathbb{R}_{\geq 0}$, called a *metric*, such that

- $d(x, y) = d(y, x)$ for all $x, y \in X$;
- if $x, y \in X$ then $d(x, y) = 0$ iff $x = y$;
- for all $x, y, z \in X$ the *triangle inequality* holds: $d(x, z) \leq d(x, y) + d(y, z)$.

In the lectures we saw that

$$d((x_1, \dots, x_n), (y_1 \dots, y_n)) = \sqrt{\sum (x_i - y_i)^2}$$

is a metric on $\mathbb{R}^n$.

If $(X, d)$ is a metric space and $x_0 \in X$ then we define the *closed ball*

$$B(x, r) = \{x \in X \mid d(x, x_0) \leq r\}$$

and the *open ball*

$$U(x, r) = \{x \in X \mid d(x, x_0) < r\}$$

of radius $r \in \mathbb{R}_{>0}$ centered at $x$. We will often call balls of radius 1 *unit balls*.

A subset $U$ of $X$ is open iff for all $x \in U$ there is an $r \in \mathbb{R}_{>0}$ such that $U(x, r) \subset U$. This gives a topology on $X$ which is called the *topology induced by the metric d*. As we shall see, it often happens that different metrics induce the same topology.

We will often simply write $X$ instead of $(X, d)$.

**Question 1.** Let $X$ be a metric space.
(a) Show that $U(x, r)$ is open and $B(x, r)$ is closed for all $x \in X$ and $r \in \mathbb{R}_{>0}$.
(b) Is it always true (i.e., for all $X, x, r$) that the closure of $U(x, r)$ is $B(x, r)$?

**Question 2.** In the lectures we have defined the following metrics on $\mathbb{R}^n$: if $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n)$ then

$$d_1(x, y) = \sum |x_i - y_i|, d_2(x, y) = \sqrt{\sum (x_i - y_i)^2}, d_\infty(x, y) = \max(|x_i - y_i|).$$

Let $U_1(x, r), U_2(x, r), U_\infty(x, r)$ be the corresponding open balls centered at $x \in \mathbb{R}^n$. Draw them for $n = 2, r = 1$. Show that for each $r_1 \in \mathbb{R}_{>0}$ and every $x \in \mathbb{R}^n$ there are $r_2, r_\infty, r'_1$ such that

$$U_1(r_1, x) \supset U_2(r_2, x) \supset U_\infty(r_\infty, x) \supset U_1(r'_1, x).$$

Deduce that all three metrics induce the same topology on $\mathbb{R}^n$.

**Isometries**

A map $f : X \to Y$ between two metric spaces is an *isometry* iff it is distance preserving, i.e., iff for all $x_1, x_2 \in X$ we have $d(x_1, x_2) = d(f(x_1), f(x_2))$. Note that an isometry needn't be bijective: take e.g. the map $[0, \infty) \to [0, \infty)$ given by $x \mapsto x + 1$. If there is a bijective isometry $X \to Y$ we say that $X$ and $Y$ are *isometric*.

**Question 3.** Show that it is possible for the intersection of two closed balls with respect to the metrics $d_1$ and $d_\infty$ to be again a closed ball with respect to the same metric. Deduce that none of the metric spaces $(\mathbb{R}^n, d_1)$ and $(\mathbb{R}^n, d_\infty)$ is isometric to $(\mathbb{R}^n, d_2)$.

**Question 4.** a) Show that $(\mathbb{R}^2, d_1)$ and $(\mathbb{R}^2, d_\infty)$ are isometric. [Hint: draw the unit balls and try to find a map that takes the unit balls of one metric to those of the other.]

b) Show that for $n \geq 3$ the spaces $(\mathbb{R}^n, d_1)$ and $(\mathbb{R}^n, d_\infty)$ are not isometric. [Hint: given a closed unit ball $B$ with respect to one of these metrics, how many closed unit balls $B'$ are there such that $B \cap B'$ is a point?]

**Question 5.** Show that any isometry of $(\mathbb{R}^n, d_2)$ can be written as $x \mapsto Ax + v$ where $v \in \mathbb{R}^n$ and $A$ is an orthogonal $n \times n$-matrix. [Hint: find a characterisation for lines and hyperplanes in terms of the metric, i.e. without referring to the structure of a vector space.]

## Completeness

Let $(x_n)$ be a sequence of elements of a metric space $(X,d)$. We say that $x \in X$ is the *limit* of $(x_n)$ and we write $x = \lim_{n\to\infty} x_n$ iff for all $\varepsilon > 0$ there is an $M \in \mathbb{Z}_{\geq 0}$ such that for all $n > M$ the distance $d(x, x_n) < \varepsilon$. Equivalently, for any open $U \ni x$ there is an $M \in \mathbb{Z}_{\geq 0}$ such that for all $n > M$ we have $x_n \in U$. Whether or not $x = \lim_{n\to\infty} x_n$, depends only on the topology of $X$, not on the metric.

Recall that $(x)_n$ is *Cauchy* iff for all $\varepsilon > 0$ there is an $N \in \mathbb{Z}_{\geq 0}$ such that for any $m, n \geq N$ the distance $d(x_n, x_m) < \varepsilon$. Whether or not $(x_n)$ is Cauchy depends on the metric, not just on the topology.

We say that $(X,d)$ is *complete* iff any Cauchy sequence $(x_n)$ in it has a limit. Whether or not $(X,d)$ is complete depends on the metric as well.

We assume that $\mathbb{R}$ with the usual metric $d(x,y) = |x-y|$ is complete.

**Question 6.** a) Show that $\mathbb{R}^n$ equipped with either of the metrics $d_1, d_2, d_\infty$ is complete.

b) Set $\mathbb{R}^\infty$ to be the set of all sequences $(x_1, x_2, \ldots)$ of real numbers all but finitely many of whose terms are zero. As above, we can equip $\mathbb{R}^\infty$ with the metrics $d_1, d_2, d_\infty$. Show that the resulting metric spaces are not complete.

**Question 7.** Construct a metric on $(0,1)$ that induces the usual topology on it and that turns it into a complete metric space. Check that the sequence $(x_n)$ with $x_n = 1/n, n \in \mathbb{Z}_{>0}$ is not Cauchy.

## Problems for discussion

For a topological space $X$ set $C^b(X,\mathbb{R})$ to be the space of all real-valued bounded continuous functions $X \to \mathbb{R}$. This is a vector space with respect to pointwise addition and multiplication by real numbers. Define a norm on this vector space by setting $||f|| = \sup_{x\in X} |f(x)|$. Note that in the case $X = \{1, 2, \ldots, n\}$ with the discrete topology the space $C^b(X,\mathbb{R})$ is nothing but $\mathbb{R}^n$ and the norm is the one that we used above to construct the metric $d_\infty$.

**1.** Show that $C^b(X,\mathbb{R})$ is complete.

In a similar way one defines $C^b(X,V)$ where $V = \mathbb{R}^n$ with the usual metric $d_2$, or more generally, any real vector space with a norm that induces a complete metric (such vector spaces are called *Banach*). The same argument shows that the resulting spaces are complete.

**2.** Now we construct a family of functions $f_n : [0,1] \to [0,1]^2, n \in \mathbb{Z}_{\geq 0}$. The 0-th one, $f_0$, is shown on the blackboard. Suppose we have already constructed $f_n$ and that it takes the $i$-th segment $I_i$ of length $\frac{1}{9^n}$ to one of the squares, call it $Q_i$, obtained by cutting $[0,1]^2$ into $9^n$ equal squares. Then $f_{n+1}$ will take $I_i$ again to $Q_i$ but in such a way that the resulting curve visits the centres of each of the 9 small equal squares that $Q_i$ can be cut into. For example, we can use a smaller copy of $f_0$.

Show that $d(f_n(t), f_{n+m}(t)) \leq \frac{\sqrt{2}}{9^m}$ where $n, m \in \mathbb{Z}_{>0}$ and $d$ is the $d_2$ metric on the square. Deduce that $(f_n)$ converges to a continuous function $f : [0,1] \to [0,1]^2$.

**3.** Shor that every $(x,y) \in [0,1]^2$ is $\leq \frac{1}{9^n\sqrt{2}}$ away from a point in the image of $f_{m+n}, m, n \in \mathbb{Z}_{\geq 0}$. Deduce using $f = \lim_{n\to\infty} f_n$ that $(x,y)$ is in the closure of the image of $f$.

**4.** Show that $f$ is surjective. Is it injective?

In a similar way one constructs continuous surjective mappings $[0,1] \to [0,1]^k$ and $\mathbb{R} \to \mathbb{R}^k$ for all $k \in \mathbb{Z}_{\geq 0}$. Such maps are called *space filling curves*. It's not that one encounters such spaces very often in topology (one doesn't) but their existence shows that statements like "$\mathbb{R}^n$ is not homeomorphic to $\mathbb{R}^m$ unless $m = n$" or "there is no continuous injective map $\mathbb{R}^n \to \mathbb{R}^m$ if $n > m$", which are intuitively clear, should nevertheless not be taken for granted.