LECTURE I

1. Minkowski space-time metric is as follows:

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = dt^2 - d\vec{x}^2. \tag{1}$$

Throughout these lectures we set the speed of light to one $c = 1$, unless otherwise stated. Here $\mu, \nu = 0, \ldots, 3$ and Minkowskian metric tensor is

$$||\eta_{\mu\nu}|| = Diag\,(1, -1, -1, -1)\,. \tag{2}$$

The bilinear form defining the metric tensor is invariant under the hyperbolic rotations:

$$\begin{aligned} t' &= t\cosh\alpha + x\sinh\alpha, \\ x' &= t\sinh\alpha + x\cosh\alpha, \\ \alpha = const, &\quad y' = y, \quad z' = z, \end{aligned} \tag{3}$$

i.e. $dt^2 - d\vec{x}^2 = (dt')^2 - (d\vec{x}')^2$.

This is the so called Lorentz boost, where $\cosh\alpha = \gamma = 1/\sqrt{1 - v^2}$, $\sinh\alpha = v\,\gamma$. Its physical meaning is the transformation from an *inertial* reference system to another *inertial* reference system. The latter one moves along the x axis with the constant velocity v with respect to the initial reference system.

Under an arbitrary coordinate transformation (not necessarily linear), $x^\mu = x^\mu\left(\bar{x}^\nu\right)$, the metric can change in an unrecognizable way, if it is transformed as the second rang tensor (see the next lecture):

$$g_{\alpha\beta}\left(\bar{x}\right) = \eta_{\mu\nu}\, \frac{\partial x^\mu}{\partial \bar{x}^\alpha}\, \frac{\partial x^\nu}{\partial \bar{x}^\alpha}. \tag{4}$$

But it is important to note that, as the consequence of this transformation of the metric, the interval does not change under such a change of coordinates:

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = g_{\alpha\beta}\left(\bar{x}\right) d\bar{x}^\alpha d\bar{x}^\beta. \tag{5}$$

In fact, it is natural to expect that if one has a space–time, then the distance between any its two–points does not depend on the way one draws the coordinate lattice on it. (The lattice is obtained by drawing three–dimensional hypersurfaces of constant coordinates $\bar{x}^\mu$ for each $\mu = 0, \ldots, 3$ with fixed lattice spacing in every direction.) Also it is natural to expect that the laws of physics should not depend on the choice of the coordinates in the space–time. This axiom is referred to as general covariance and is the basis of the General Theory of Relativity.

2. Lorentz transformations in Minkowski space–time have the meaning of transitions between inertial reference systems. Then what is a meaning of an arbitrary coordinate transformation? To answer this question let us start with the transition into a non–inertial reference system in Minkowski space–time.

The simplest non–inertial motion is the one with the constant linear acceleration. Three–acceleration cannot be constant in a relativistic situation. Hence, we consider a motion of a particle with a constant four–acceleration, $w^\mu \, w_\mu = -a^2 = const$, where $w^\mu = d^2 z^\mu(s)/ds^2$ and $z^\mu(s) = \left[z^0(s), \vec{z}(s)\right]$ is the world–line of the particle parametrized by the proper time s. Let us choose the spatial reference system such that the acceleration will be directed along the first axis. Then we have that:

$$\left(\frac{d^2 z^0}{ds^2}\right)^2 - \left(\frac{d^2 z^1}{ds^2}\right)^2 = -a^2. \tag{6}$$

Thus, the components of the four–acceleration compose a hyperbola. Hence, the standard solution of this equation is as follows:

$$z^0(s) = \frac{1}{a}\,\sinh(a\, s), \quad z^1(s) = \frac{1}{a}\left[\cosh(a\, s) - 1\right]. \tag{7}$$

The integration constant in $z^1(s)$ is chosen for the future convenience.

Thus, one has the following relation between z^1 and z^0 themselves:

$$\left(z^1 + \frac{1}{a}\right)^2 - \left(z^0\right)^2 = \frac{1}{a^2}. \tag{8}$$

I.e. the world–line of a particle which moves with constant eternal acceleration is just a hyperbola (see fig. (1)). Note that the three–dimensional part of the acceleration is always along the positive direction of the x axis. Hence, for the negative s the particle is actually decelerating, while for the positive s it accelerates. (We assume that $s = 0$ corresponds to $t = 0$, as is shown on the fig. (1).) The asymptotes of the hyperbola are the light–like lines, $z^1 = \pm z^0 - 1/a$. Hence, even if one moves with eternal constant acceleration, he cannot exceed the speed of light.

Moreover, for small $a\, z^0$ we find from (8) that: $z^1 \approx a\left(z^0\right)^2/2$. In fact, for small proper times, $a\, s \ll 1$ we have that $z^0 \approx s$, $v \approx dz^1/ds \approx a\, z^0 \ll 1$ and obtain the standard nonrelativistic acceleration, which, however, gets modified according to (8) once the particle reaches high enough velocities. It is important to stress at this point that eternal constant acceleration is physically impossible due to infinite energy consumption. I.e. here we are just discussing some mathematical abstraction, which, however, is helpful to clarify some important issues.

These observations will allow us to find the appropriate coordinate system for accelerated observers. The motion with a constant eternal acceleration is homogeneous, i.e. accelerated observer cannot distinguish any moment of its proper time from any other. Hence, it is natural to expect that there should be static (invariant under both time–translations and time–reversal transformations) reference frame seen by accelerated observers. Inspired by (7), we propose the following coordinate change:

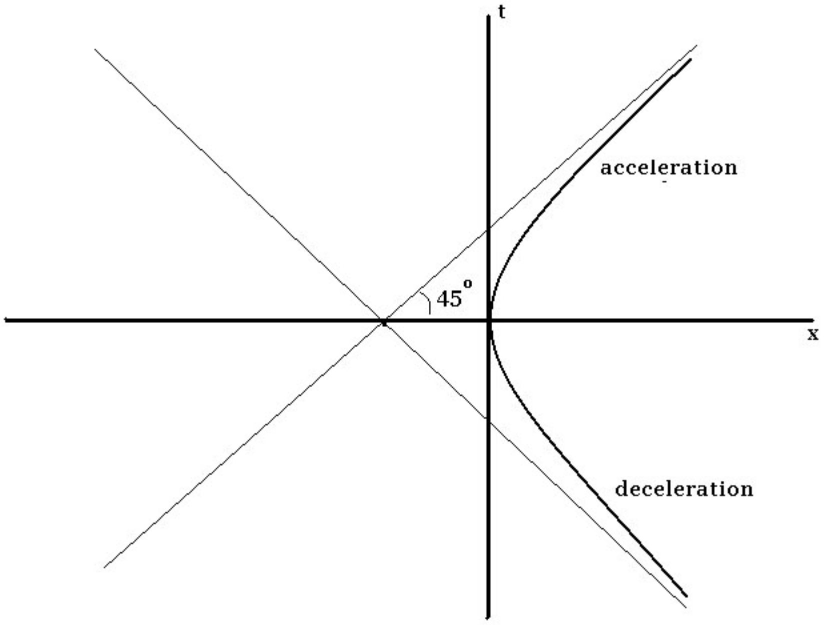

Figure 1: In this picture and also in the other pictures of this lecture we show only slices of fixed y and z.

$$t = \rho \sinh\tau, \quad x = \rho \cosh\tau, \quad \rho \geq 0,$$
$$y' = y, \quad \text{and} \quad z' = z. \tag{9}$$

Please note that these coordinates cover only quoter of the entire Minkowski space. Namely — the right quadrant. In fact, since $\cosh\tau \geq |\sinh\tau|$, we have that $x \geq |t|$. It is not hard to guess the coordinates which will cover the left quadrant. For that one has to choose $\rho \leq 0$ in (9).

Under such a coordinate change we have:

$$dt = d\rho \sinh\tau + \rho\, d\tau \cosh\tau, \quad dx = d\rho \cosh\tau + \rho\, d\tau \sinh\tau. \tag{10}$$

Then $dt^2 - dx^2 = \rho^2\, \tau^2 - d\rho^2$ and:

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 = \rho^2\, d\tau^2 - d\rho^2 - dy^2 - dz^2 \tag{11}$$

is the so called <u>Rindler metric</u>. It is not constant, $||g_{\mu\nu}|| = Diag(\rho^2, -1, -1, -1)$, but is time–independent and diagonal (static), as we have expected.

In this metric the levels of the constant coordinate time τ are straight lines $t/x = \tanh\tau$ in the $x - t$ plane (or three–dimensional flat planes in the entire Minkowski space). The levels of the constant ρ are the hyperbolas $x^2 - t^2 = \rho^2$ in the $x - t$ plane. The latter ones correspond to world–lines of observers which are moving with constant four–accelerations equal to $1/\rho$ on a slice of fixed

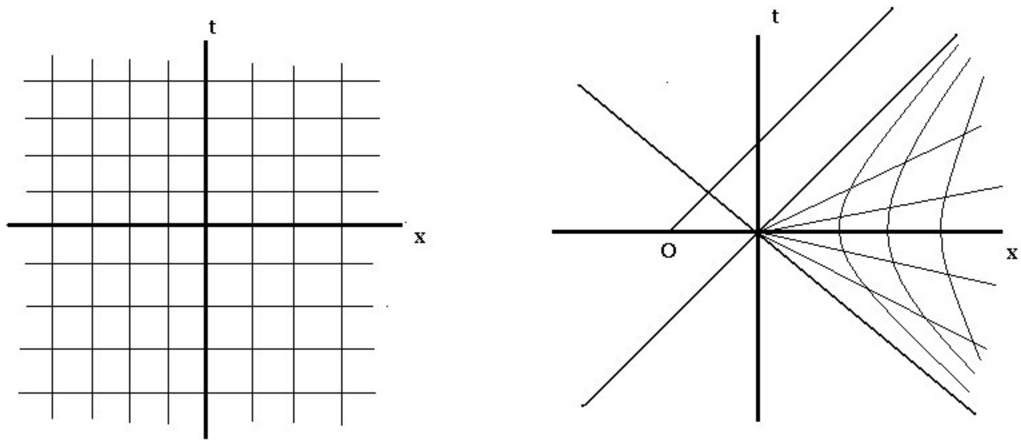

Figure 2:

y and z. The hyperbolas degenerate to light-like lines $x = \pm t$ as $\rho \to 0$. These are asymptotes of the hyperbolas for all ρ. As one takes ρ closer and closer to zero the corresponding hyperbolas are closer and closer to their asymptotes. Note also that $\tau = -\infty$ corresponds to $x = -t$ and $\tau = +\infty$ — to $x = t$. As the result we get a change of the coordinate lattice, which is depicted on the fig. (2).

3. The important feature of the Rindler's metric (11) is that it degenerates at $\rho = 0$. This singularity is the so called coordinate singularity. It is similar to the singularity of the polar coordinates $dr^2 + r^2 d\varphi^2$ at $r = 0$. The space–time itself is regular at $\rho = 0$. It is just flat Minkowski space–time at the light–like lines $x = \pm t$. Another important feature of the Rindler's metric is that the speed of light is coordinate dependent:

$$ds^2 = 0, \quad \text{then} \quad \left|\frac{d\rho}{d\tau}\right| = \rho, \quad \text{if} \quad dy = dz = 0. \tag{12}$$

At the same time, in the proper coordinates the speed of light is just equal to one $d\rho/ds = d\rho/\rho d\tau = 1$. Furthermore, as $\rho \to 0$ the speed of light, $d\rho/d\tau$, becomes zero. This phenomenon is related to the fact that if an observer starts an eternal acceleration with $a = 1/\rho$, say at the moment of time $t = 0 = \tau$, then there is a region in Minkowski space–time from which light rays cannot reach him. In fact, as shown on the fig. (2) if a light ray was emitted from a point like O it is parallel to the asymptote $x = t$ of the world–line of the observer in question. As the result, the light ray never intersects with hyperbolas, i.e. never catches up with eternally accelerating observer. These are the reasons why one cannot extend the Rindler metric beyond the light–like lines $x = \pm t$. The three–dimensional surface $x = t$ of the entire Minkowski space–time is referred

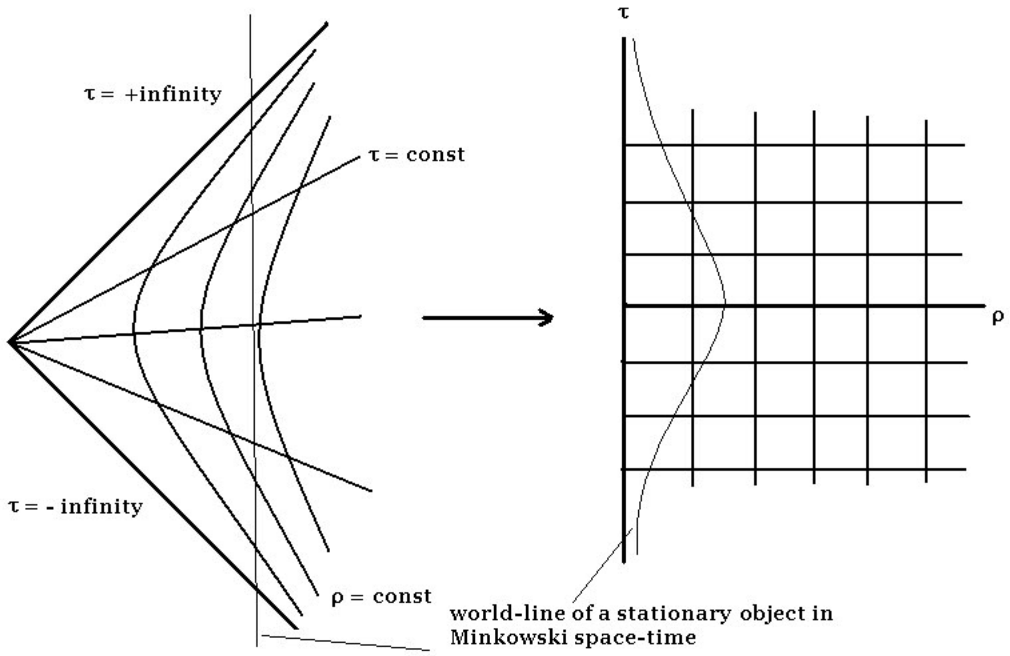

Figure 3:

to as the future event horizon of the Rindler's observers (those which are staying at the constant ρ positions throughout their entire life time). At the same time $x = -t$ is the past event horizon of the Rindler's observers.

Note that if an observer accelerates during a finite period of time, then, after the acceleration is switched off, his world–line will be a straight line, which is tangential to the corresponding hyperbola. (I.e. the observer will continue moving with the gained velocity.) The angle this tangential line has with the Minkowskian time axis is less that 45^o. Hence, sooner or later the light ray emitted from a point like O will actually reach such an observer. I.e. this observer does not have an event horizon.

Another interesting phenomenon which is seen by the Rindler's observers is shown on the fig. (3). A stationary object, $x = const$, in Minkowski space–time cannot cross the event horizon of the Rindler's observers during any finite period of the coordinate time τ. This object just slows down and only asymptotically approaches the horizon. Note that, as $\rho \to 0$ a fixed finite portion of the proper time, $ds = \rho d\tau$, corresponds to increasing portions of the coordinate time. Recall also that $\tau = -\infty$ corresponds to $x = -t$ and $\tau = +\infty$ — to $x = t$.

All these peculiarities of the Rindler metric is the price one has to pay for the consideration of the physically impossible eternal acceleration. However, if one were transferring to a reference system of observers which are moving with accelerations only during finite proper times, then he would obtain a non–stationary metric due to the inhomogeneity of such a motion.

The main lesson to draw from these observations is as follows. The physical meaning of a general coordinate transformation that mixes spatial and time coordinates is a transition to another, not necessary inertial, reference system. In this case curves corresponding to fixed spatial coordinates (e.g. $d\rho = dy = dz = 0$) are world–lines of (non–)inertial observers. As the result, the essence of the general covariance is that physical laws should not depend on the choice of observers.

4. If even in flat space–time one can choose curvilinear coordinates and obtain a non–trivial metric tensor $g_{\mu\nu}(x)$, then how can one distinguish flat space–time from the curved one? Furthermore, since we understood the physics behind the curvilinear coordinates in flat space–time, then it is also natural to ask what is the physics behind curved space–times? To start answering these questions in the following lectures let us solve here a simple problem.

Namely, let us consider a free particle moving in a space–time with the metric $g_{\mu\nu}(x)$. Let us find its world–line via the minimal action principle. If one considers a world–line $z^\mu(\tau)$ parametrized by a parameter τ (that, e.g., could be either a coordinate time or the proper one), then the simplest invariant characteristic that one can associate to the world–line is its length. Hence, the natural action for the free particle should be proportional to the length of its world–line. The reason why we are looking for an invariant action is that we expect the corresponding equations of motion to be covariant (i.e. to have the same form in all coordinate systems).

If one approximates the world–line by a broken line consisting of a chain of small intervals, then its length can be approximated by the expression as follows:

$$L = \sum_{i=1}^{N} \sqrt{g_{\mu\nu}\left[z_i\right]\, \left[z_{i+1} - z_i\right]^\mu\, \left[z_{i+1} - z_i\right]^\nu}, \tag{13}$$

which follows from the definition of the metric. In the limit $N \to \infty$ and $|z_{i+1} - z_i| \to 0$ we obtain an integral instead of the sum. As the result, the action should be as follows:

$$S = -m \int_1^2 ds = -m \int_{\tau_1}^{\tau_2} d\tau \sqrt{g_{\mu\nu}\left[z(\tau)\right]\, \dot{z}^\mu\, \dot{z}^\nu}. \tag{14}$$

Here $\dot{z} = dz/d\tau$. The coefficient of the proportionality between the action, S, and the length, L, is minus the mass, m, of the particle. This coefficient follows from the complementarity — from the fact that when $g_{\mu\nu}(x) = \eta_{\mu\nu}$ we have to obtain the standard action for the relativistic particle in the Special Theory of Relativity.

Note that the action (14) is invariant under the coordinate transformations and also under the reparametrizations, $\tau \to f(\tau)$, if one respects the ordering of points along the world–line $df/d\tau \geq 0$. In fact, then:

$$d\tau \sqrt{g_{\mu\nu} \frac{dz^\mu}{d\tau} \frac{dz^\nu}{d\tau}} = df \sqrt{g_{\mu\nu} \frac{dz^\mu}{df} \frac{dz^\nu}{df}}.$$

Let us find equations of motion that follow from the minimal action principle applied to (14). The first variation of the action is:

$$\delta S = -m \int_{\tau_1}^{\tau_2} d\tau \, \frac{\delta\left[g_{\mu\nu}(z)\, \dot{z}^\mu\, \dot{z}^\nu\right]}{2\sqrt{\dot{z}^2}} =$$

$$= -m \int_{\tau_1}^{\tau_2} \frac{d\tau\,\sqrt{\dot{z}^2}}{2\sqrt{\dot{z}^2}\,\sqrt{\dot{z}^2}} \left[\delta g_{\mu\nu}(z)\ \ \dot{z}^\mu\,\dot{z}^\nu + g_{\mu\nu}(z)\,\delta\dot{z}^\mu\,\dot{z}^\nu + g_{\mu\nu}(z)\,\dot{z}^\mu\,\delta\dot{z}^\nu\right]. \tag{15}$$

Here we denote $\dot{z}^2 = g_{\alpha\beta}\dot{z}^\alpha\dot{z}^\beta$. Using the fact that $\sqrt{\dot{z}^2}\,d\tau = \sqrt{g_{\mu\nu}\,dz^\mu\,dz^\nu} = ds$ we can change the parametrization from τ to the proper time s. After that we integrate by parts in the last two terms in the last line of (15). This way we get reed from the differentiation of δz: $\delta\dot{z} = \frac{d}{ds}\delta z$. Then, using the Dirichlet boundary conditions, i.e. assuming that $\delta z(s_1) = \delta z(s_2) = 0$, we arrive at the following expression for the first variation of S:

$$\delta S = -m \int_{s_1}^{s_2} \frac{ds}{2} \left\{\partial_\alpha g_{\mu\nu}(z)\,\delta z^\alpha\,\dot{z}^\mu\,\dot{z}^\nu - \frac{d}{ds}\left[g_{\mu\nu}(z)\ \ \dot{z}^\nu\right]\,\delta z^\mu - \frac{d}{ds}\left[g_{\mu\nu}(z)\ \ \dot{z}^\mu\right]\,\delta z^\nu\right\} =$$

$$= -m \int_{s_1}^{s_2} \frac{ds}{2} \left\{\partial_\alpha g_{\mu\nu}\delta z^\alpha\,\dot{z}^\mu\,\dot{z}^\nu - \partial_\alpha g_{\mu\nu}\,\dot{z}^\alpha\,\dot{z}^\nu\,\delta z^\mu - \partial_\alpha g_{\mu\nu}\,\dot{z}^\alpha\,\dot{z}^\mu\,\delta z^\nu - 2\,g_{\mu\nu}\ddot{z}^\mu\,\delta z^\nu\right\} =$$

$$= -m \int_{s_1}^{s_2} ds \left[\frac{1}{2}\left(\partial_\alpha\, g_{\mu\nu} - \partial_\mu g_{\alpha\nu} - \partial_\nu g_{\mu\alpha}\right)\,\dot{z}^\mu\,\dot{z}^\nu - g_{\mu\alpha}\,\ddot{z}^\mu\right]\,\delta z^\alpha. \tag{16}$$

In these expressions $\dot{z} = dz/ds$ and also we have used that $g_{\mu\nu}\,\ddot{z}^\mu\,\delta z^\nu = g_{\mu\nu}\,\delta z^\mu\,\ddot{z}^\nu$ because $g_{\mu\nu} = g_{\nu\mu}$. Taking into account that according to the minimal action principle δS should be equal to zero for any δz^α, we arrive at the following equation:

$$\ddot{z}^\mu + \Gamma^\mu_{\nu\alpha}(z)\,\dot{z}^\nu\,\dot{z}^\alpha = 0, \tag{17}$$

which is referred to as the geodesic equation. Here

$$\Gamma^\mu_{\nu\alpha} = \frac{1}{2}\,g^{\mu\beta}\left(\partial_\nu\, g_{\alpha\beta} + \partial_\alpha\, g_{\beta\nu} - \partial_\beta g_{\nu\alpha}\right) \tag{18}$$

are the so called Christoffel symbols and $g^{\mu\beta}\,g_{\beta\nu} = \delta^\mu_\nu$ is the inverse metric tensor.

Problems:

- Show that the metric $ds^2 = (1 + a\,h)^2\ d\tau - dh^2 - dy^2 - dz^2$ (homogeneous gravitational field) also covers the Rindler space–time. Find the coordinate change from this metric to the one used in the lecture.
- Find the coordinates which cover the lower and upper quadrants (complementary to those which are covered by Rindler's coordinates) of the Minkowski space–time.

- Find the coordinate transformation and the stationary (invariant only under time–translations, but not under time–reversal transformation) metric in the rotating reference system with the angular velocity ω. (See the corresponding paragraph in Landau–Lifshitz.)

- (*) Find the coordinate transformation and the stationary metric in the orbiting reference system, which moves on the radius R with the angular velocity ω.

- (*) Consider a particle which was stationary in an inertial reference system. Then its acceleration was adiabatically turned on and kept finite for long period of time. And finally its acceleration was adiabatically switched off. I.e. this particle for the beginning is stationary then accelerates for a while, and finally proceeds its motion with a constant gained velocity. Find the world–line for such a motion. Find a metric which is seen by such observers.

- (*) Find the equation for geodesics in the non–Rimanian metric:

  $$ds^n = g_{\mu_1 \ldots \mu_n} \, dx^{\mu_1} \ldots dx^{\mu_n}.$$

  (**) What kind of geometries (instead of the Minkowskian one) there are, if $g_{\mu_1 \ldots \mu_n}$ has only constant (coordinate independent) components?

- (**) What kind of geometry (instead of the Minkowskian one) there is, if $g_{\mu\nu} = Diag(1, 1, -1, -1)$ instead of Minkowskian metric?

Subjects for further study:

- Radiation of the homogeneously accelerating charges: What is the intensity seen by a distant inertial observer? What is the intensity seen by a distant co–moving non–inertial observer? What is the invariant energy loss of the homogeneously accelerating charge? Does a free falling charge in a homogeneous gravitational field create a radiation? Does a charge, which is fixed in a homogeneous gravitational field, create a radiation?

- Action and minimal action principle for strings and membranes in arbitrary dimensions.