LECTURE VI

1. In this lecture we continue the discussion of the physics behind the Schwarzschild geometry. Namely we find here the so called Oppenheimer–Snyder solution, which describes the collapse process of a spherically symmetric, non–rotating body into the black hole.

Consider a static star (a ball filled with a matter) surrounded by empty space. We will derive the metric describing such a situation in the next lecture. The star is static due to an internal pressure until some moment of time $t = 0$ and then the pressure is switched off. Say before $t = 0$ there were some thermonuclear processes inside this star, which were producing the internal pressure. But by the moment $t = 0$ the entire thermonuclear fuel was used out.

To model the collapse process after $t = 0$ we assume that inside the star there is a homogeneous *pressureless* dust. Also we assume that the original star was ideal ball with ideal spherical surface and that the collapse process goes in such a way that homogeneity of the matter inside the star and the spherical symmetry is never violated. This is a highly unstable situation because any its perturbation violating these symmetries will grow in time due to the tidal forces. We neglect such perturbations.

Thus, inside the ball the energy momentum tensor is $T_{\mu\nu} = \rho(\tau) u_\mu u_\nu$, where the density $\rho(\tau)$ is just a function of the proper time τ because of the spatial homogeneity. Outside the ball we have vacuum, $T_{\mu\nu} = 0$.

Such a massive ball of matter should create a spherically symmetric and asymptotically flat gravitational field in vacuum. Thus, due to Birkhoff theorem outside the ball the metric has to be the Schwarzschild one. Any violation of the spherical symmetry will lead to time–dependent gravitational fields in vacuum, i.e. to a creation of gravitational waves, which are discussed in the lectures IX and X. Neglecting such processes is exactly the approximation that we use here.

To give an intuitive explanation why spherically symmetric collapse does not create gravitational radiation, let us discuss the following situation. Consider a ball which is electrically charged and the charge is homogeneously distributed over its volume. Suppose now that for some reason this ball rapidly shrinks in such a way that the homogeneity and spherical symmetry are respected. It is not hard to see that independently of the radius of the ball it creates the same Coulomb field outside itself. This is related to the uniqueness of the Coulomb solution of the Maxwell equations, which was mentioned in the lecture IV.

Thus, the magnetic field outside the ball is vanishing and such an accelerated motion of the charge does not create an electromagnetic radiation. The point is that to have a radiation there should be at least dipole moment, which is changing in time, while in the case of the ball all momenta are zero with respect to its center. It happens that to have a gravitational radiation, as we will see in the lectures IX and X, there should be even quadruple moment, which changes in time. Dipole moment is not enough. Also from the Newton's gravitation we know that an ideal spherical massive ball creates the same potential outside itself independently of its radius. The form of the potential just depends on the mass of the ball. In general relativity the situation is similar due to the Birkhoff's theorem.

2. All in all, the metric outside the ball is

$$ds_+^2 = \left(1 - \frac{r_g}{r}\right) dt^2 - \frac{dr^2}{1 - \frac{r_g}{r}} - r^2 \, d\Omega^2. \tag{1}$$

Here $r_g = 2\kappa M$, where M is the mass of the ball, which remains constant during the collapse process, because of the absence of the radiation. Intuitively it should be clear that then there is no energy which fluxes away to infinity with the radiation and the energy of the ball remains constant.

This metric is valid outside of the surface of the ball Σ, which radially shrinks down during the collapse process. Then the world–hypersurface of Σ is $z^\mu(\tau) = [T(\tau), R(\tau)]$ and at every given time slice it is an ideal sphere. (Note that in this lecture we denote by the same Σ both the three–dimensional world–hypersurface and its two–dimensional time–slices.) Hence, Σ occupies all the values of the spherical angles θ and φ. Here $R(\tau)$ decreases as the proper time τ goes by. The initial value of the $R(\tau_0) = R_0$ is grater than r_g. Otherwise we would have had a black hole rather than a star at the initial stage.

Inside the ball we have a spatially homogeneous metric whose time–slices are compact and are decreasing in size as time goes by. The suitable metric is:

$$ds_-^2 = d\tau^2 - a^2(\tau) \, \left[d\chi^2 + \sin^2\chi \, d\Omega^2\right], \quad d\Omega^2 = d\theta^2 + \sin^2\theta \, d\varphi^2. \tag{2}$$

We discuss the physics and the origin of such metrics in the lecture XI in grater details. Its spatial section $d\tau = 0$ is the three–sphere, whose metric is $a^2(\tau) \, \left[d\chi^2 + \sin^2\chi \, d\Omega^2\right]$, $\chi \in [0, \pi]$. The radius of the three–sphere $a(\tau)$ is decreasing as the time τ goes by.

Thus, we have

$$g_{\tau\tau} = 1, \quad g_{\chi\chi} = -a^2, \quad g_{\theta\theta} = -a^2 \, \sin^2\chi, \quad \text{and} \quad g_{\varphi\varphi} = -a^2 \, \sin^2\chi \, \sin^2\theta. \tag{3}$$

The resulting non–zero components of the Christoffel symbols are $\Gamma^0_{ij} = \dot{a} \, a \tilde{g}_{ij}$ and $\Gamma^i_{0j} = \frac{\dot{a}}{a} \, \delta^i_j$. Here $\dot{a} \equiv da/d\tau$, $i = 1, 2, 3$ and $\tilde{g}_{ij}$ is the metric of the three–sphere of unit radius. As the result also Γ^i_{jk} components of the Christophel symbols are not zero and are proportional to those of the three–sphere, but we do not need their explicit form in this lecture.

One can find from these Christoffel symbols that the "00" part of the Einstein tensor, $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} \, g_{\mu\nu} \, R$, has the following form:

$$G_{00} = \frac{3}{a^2} \, \left(\dot{a}^2 + 1\right). \tag{4}$$

At the same time the energy–momentum tensor inside the ball is $T_{\mu\nu} = \rho(\tau) \, u_\mu \, u_\nu$. In the reference frame of (3) the dust remains stationary, i.e. $d\chi = d\theta = d\varphi = 0$. Hence, $u_\mu = (1, 0, 0, 0)$ and the only non–zero component of $T_{\mu\nu}$ is $T_{00} = \rho(\tau)$. As the result from (4) we find that one of the Einstein equations is as follows:

$$\dot{a}^2 + 1 = \frac{8\pi\kappa}{3}\,\rho\, a^2. \tag{5}$$

Equations for T_{0j} do *not* lead to non–trivial relations: they just give relations stating that $0 = 0$. As we have explained at the end of the lecture III, instead of some the remaining equations one can use the energy–momentum tensor conservation condition,

$$0 = D_\mu T^\mu_\nu = \partial_\mu \left(\rho\, u^\mu\, u_\nu\right) + \Gamma^\mu_{\beta\mu}\,\rho\, u^\beta\, u_\nu - \Gamma^\beta_{\nu\mu}\,\rho\, u^\mu\, u_\beta. \tag{6}$$

Here we have four equations — one for each value of ν. If $\nu = 1, 2, 3$ then again we get trivial relations $0 = 0$. However, if $\nu = 0$ we obtain the following equation:

$$\dot{\rho} + 3\,\frac{\dot{a}}{a}\,\rho = 0, \quad \text{where} \quad \dot{\rho} = \frac{d\rho}{d\tau}. \tag{7}$$

Hence, $\partial_\tau \left(\rho\, a^3\right) = 0$ and we obtain the obvious result that if the volume of spatial sections decreases as $a^3(\tau)$ the density of the dust is increasing as $\rho(\tau) = \frac{const}{a^3(\tau)}$. Let us choose here such a constant that $\rho\, a^3 = \frac{3}{8\pi\kappa}\, a_0$. Then the solution of (5) has the following parameterized form:

$$a(\eta) = \frac{a_0}{2}\,\left(1 + \cos\eta\right), \quad \text{and} \quad \tau(\eta) = \frac{a_0}{2}\,\left(\eta + \sin\eta\right). \tag{8}$$

We discuss the physical meaning of the parameter η and of such solutions in the lecture XI in grater details. Here we just check that (8) indeed solves (5). In fact, $\frac{d\tau}{d\eta} = \frac{a_0}{2}\,(1 + \cos\eta) = a(\eta)$. Then, $\frac{da}{d\tau} = \frac{da}{d\eta}\frac{d\eta}{d\tau} = \frac{da}{d\eta}\frac{1}{a}$. As the result from (5) we derive the equation

$$\left(\frac{da}{d\eta}\right)^2 + a^2 = a_0\, a. \tag{9}$$

One can check by explicit substitution that $a(\eta)$ from (8) solves this equation.

From (8) we see that the collapse starts at $\eta = 0$, which corresponds to $\tau = 0$. In Schwarzschild coordinates this corresponds to $t = 0$, when thermonuclear fuel inside the ball was completely spent. At this moment $a = a_0$. The collapse ends as $a \to 0$. That happens as $\eta \to \pi$, i.e. when the proper time reaches $\tau = \frac{\pi a_0}{2}$. Thus, the collapse process takes the finite proper time: the shell crosses the horizon $r = r_g$ and even reaches the singularity within the finite proper time.

3. What remains to be done is to glue the metrics ds^2_+ and ds^2_- and their first derivatives across the surface Σ of the ball. Their second derivatives, which are related to the Ricci tensor, are fixed by Einstein equations.

In terms of the metric ds^2_- the surface Σ is just a two sphere at some value χ_0 of the angle χ. Then the induced metric on the surface is

$$ds^2_-\big|_\Sigma = d\tau^2 - a^2(\tau)\sin^2\,\chi_0\, d\Omega^2. \tag{10}$$

This metric has to be related to the one on the world–hypersurface $z^\mu(\tau) = [T(\tau),\, R(\tau)]$ in the Schwarzschild background:

$$ds_+^2\big|_\Sigma = \left[\left(1 - \frac{r_g}{R}\right)\left(\frac{dT}{d\tau}\right)^2 - \frac{1}{1-\frac{r_g}{R}}\left(\frac{dR}{d\tau}\right)^2\right] d\tau^2 - R^2(\tau)\, d\Omega^2. \qquad (11)$$

To have that $ds_-^2\big|_\Sigma = ds_+^2\big|_\Sigma$, there should be relations as follows:

$$\begin{gathered} R(\tau) = a(\tau)\, \sin\chi_0, \\ \left(1 - \frac{r_g}{R}\right)\left(\frac{dT}{d\tau}\right)^2 - \frac{1}{1-\frac{r_g}{R}}\left(\frac{dR}{d\tau}\right)^2 = 1. \end{gathered} \qquad (12)$$

These relations allow us to find $R(\tau)$ from (8) and $T(\tau)$ from the equation:

$$\dot{T} = \frac{\sqrt{\dot{R}^2 + 1 - \frac{r_g}{R}}}{1 - \frac{r_g}{R}}, \quad \text{where} \quad \dot{T} = \frac{dT}{d\tau}, \quad \text{and} \quad \dot{R} = \frac{dR}{d\tau}. \qquad (13)$$

This is the world–hypersurface of the boundary of the ball as it is seen by outside observers.

From the last equation one can see that as $R \to r_g$ in the collapse process, we can neglect $1 - \frac{r_g}{R}$ in comparison with $\dot{R}^2$ under the square root. Then,

$$dT \approx -\frac{dR}{1 - \frac{r_g}{R}}, \qquad (14)$$

and the minus sign appears here because during the collapse process we have that $dR < 0$, while $dT > 0$. Thus, one obtains that

$$R(T) \approx r_g\left(1 + e^{-\frac{T}{r_g}}\right), \qquad (15)$$

i.e. from the point of view of an observer, which is fixed at some $r > R(\tau)$, the fall of the star's matter through the surface $r = r_g$ never happens. The matter of the star just asymptotically approaches its gravitational radius as $t \to +\infty$. That is true although the star falls behind its gravitational radius within the above mentioned finite *proper* time.

The next step is to glue the first derivatives of ds_-^2 and ds_+^2 across Σ. This demands some straightforward calculations with the use of differential geometry for surfaces in curved space–times. This goes beyond the scope of our lectures. But the result of the calculation is very simple and can be predicted on general physical grounds. In fact, from the gluing conditions in question one finds that

$$r_g = 2\,\kappa\, \frac{4\pi}{3}\, \rho(\tau)\, R^3(\tau) = const. \qquad (16)$$

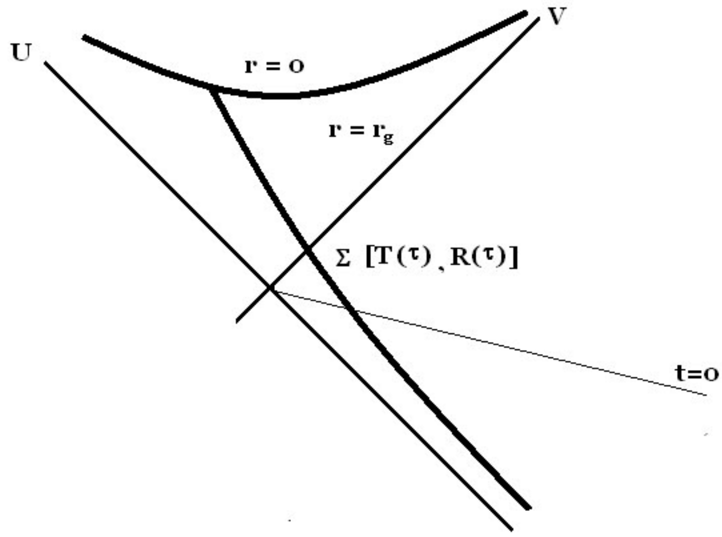

Figure 1:

Which just means that the mass of the star/black hole remains constant during the ideal spherical collapse. Moreover, the mass is appropriately related to the gravitational radius, r_g, of the Schwarzschild geometry.

4. Let us draw the Penrose–Carter diagram for the Oppenheimer–Snyder collapsing solution. To do that we have to draw separately the diagrams for ds_+^2 and ds_-^2 and to glue them across Σ. For the Schwarzschild part, ds_+^2, the diagram is shown on the fig. (1). This is just the same diagram as in the last lecture, but it is valid only beyond Σ whose world–surface is $[T(\tau), R(\tau)]$ after $t = 0$. Before $t = 0$ the surface Σ remains stationary at some radius $r = R_0 > r_g$.

To draw the Penrose–Carter diagram for ds_-^2 let us represent this metric as follows:

$$ds_-^2 = d\tau^2 - a^2(\tau)\,\left[d\chi^2 + \sin^2\chi\, d\Omega^2\right] = a^2(\eta)\,\left[d\eta^2 - d\chi^2 - \sin^2\chi\, d\Omega^2\right], \tag{17}$$

where $\eta \in [0, \pi]$ is defined in (8). Now we should drop off the conformal factor $a^2(\eta)$ and choose the relevant two–dimensional part i.e. (η, χ). Then we obtain just a part of the square for $0 \le \chi \le \chi_0$ and $0 \le \eta \le \pi$. This has to be glued to the diagram for the fig. (1) after $t = 0$. Before this moment of time the Penrose–Carter diagram of the space–time behind Σ is just very similar to a part of the Minkowski space–time diagram, as will become clear in the next lecture.

All in all, this way one finds the total Penrose–Carter diagram for the Oppenheimer–Snyder collapsing solution which is shown on the fig. (2). It can be adjusted to the shown here form by a suitable conformal transformation. It is worth stressing at this point that in doing this gluing of diagrams we drop off different conformal factors for different parts of the diagram.

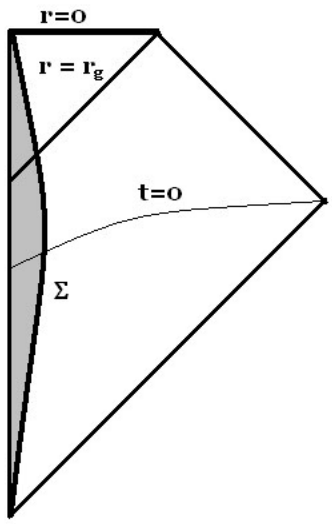

Figure 2:

As one can see, the obtained diagram does not contain the white hole part. But it is not hard to find the time reversal of the collapsing solution. It is given by the same equation as (8) for $\eta \in [\pi, 2\pi]$ or by

$$a(\eta) = \frac{a_0}{2}\,(1 - \cos\eta)\,, \quad \text{and} \quad \tau(\eta) = \frac{a_0}{2}\,(\eta - \sin\eta)\,, \tag{18}$$

if one chooses $\eta \in [0, \pi]$. This solution describes explosion starting from $\eta = 0$, when $\tau = 0$ and $a = 0$. Then as η reaches π the conformal factor inflates to $a = a_0$. If something stops the explosion at $a = a_0$, then the corresponding Penrose–Carter diagram is just the time reversal of the one shown on the fig. (2)

5. Let us see now what happens with waves which are created in the vicinity of a collapsing body as the time goes by and the body approaches its Schwarzschild radius. The arguments, which are presented here, are borrowed from the book of I.Kriplovich "General Relativity". Consider an electromagnetic excitation, which is created at a radius $r_0 = r_g + \epsilon$ in the vicinity of the horizon. Radial propagation time of this excitation from r_0 to $r \gg r_g$ follows from the equation:

$$0 = ds^2 = \left(1 - \frac{r_g}{r}\right) dt^2 - \frac{dr^2}{1 - \frac{r_g}{r}}, \tag{19}$$

and is equal to

$$t = \int_{r_0}^{r} \frac{dr}{1 - \frac{r_g}{r}} = r - r_0 + r_g \log\frac{r - r_g}{r_0 - r_g} \approx r + r_g \log\frac{r}{\epsilon}. \tag{20}$$

If the frequency at r_0 is equal to ω_0, then, as follows from the last equation of the previous lecture, at $r > r_0$ the frequency is reduced to

$$\omega = \omega_0 \sqrt{\frac{g_{00}(r_0)}{g_{00}(r)}} \approx \omega_0 \sqrt{\frac{\epsilon}{r_g}}, \quad \text{if} \quad r \gg r_g. \tag{21}$$

This effect can be understood on general physical grounds. In fact, a photon performs a work to climb out from the gravitational attraction of the massive center. Then its energy and, hence, frequency should reduce.

As follows from (20)

$$\frac{\epsilon}{r} = e^{-\frac{t-r}{r_g}}. \tag{22}$$

Hence, the frequency of the excitation depends on time as follows:

$$\omega(t) = \omega_0 \sqrt{\frac{r}{r_g}}\, e^{-\frac{t-r}{2r_g}}. \tag{23}$$

Furthermore, the phase of the electromagnetic excitation is changing as

$$\int_0^t dt' \omega(t') = -2\,\omega_0\, \sqrt{r\, r_g}\, e^{-\frac{t-r}{2\, r_g}}. \tag{24}$$

Then the spectrum of the corresponding wave–packet at large distances is as follows:

$$f(\omega) \sim \int_0^\infty dt\, e^{i\,\omega\, t} \, \exp\left[-2\, i\, \omega_0\, \sqrt{r\, r_g}\, e^{-\frac{t-r}{2\, r_g}}\right] \sim \left(2\,\omega_0\,\sqrt{r\, r_g}\right)^{2\, i\,\omega\, r_g}\, e^{-\pi\,\omega\, r_g}\, \Gamma\left(-2\, i\,\omega\, r_g\right), \tag{25}$$

where $\Gamma(x)$ is the Γ–function. Here we have dropped off the factors that do not depend on ω. As the result, the spectral density of any wave–like excitation that was created in the vicinity of the horizon is given by:

$$|f(\omega)|^2 \sim e^{-2\,\pi\,\omega\, r_g}\, \left|\Gamma\left(-2\, i\,\omega\, r_g\right)\right|^2 = \frac{\pi}{\omega\, r_g}\, \frac{1}{e^{4\,\pi\,\omega\, r_g} - 1} \approx \frac{\pi}{\omega\, r_g}\, e^{-4\,\pi\,\omega\, r_g}. \tag{26}$$

This calculation is absolutely classical. However, if in the last expression we multiply ω by $\hbar$, to obtain energy, then the exponent will acquire the form of the Boltzman's thermal factor:

$$e^{-\frac{\hbar\,\omega}{T}}, \tag{27}$$

where $T = \frac{\hbar}{4\,\pi\, r_g}$ is the so called Hawking's temperature. Hawking effect means that black holes are decaying via creation of particles with thermal spectrum. Formally, Hawking radiation of black holes follows from similar equations. However, conceptually the effect in question is much more complicated and its discussion goes beyond the scope of our lectures. But let us just point out that Hawking effect follows from a change of the ground state of quantum fields due to a collapse process.

6. Now it is worth asking the following question. Does actually the creation of the black hole ever happens from the point of view of those who always stay outside it? Or do they just see such an eternal asymptotically slowing down process which was described above? Although this is just an academic question, because there seems to be no device which will be always sensitive to the exponentially suppressed factor e^{-t/r_g} in (23) or in (15), as $t \to +\infty$, we still would like to address it here. I think that an answer on this question may be relevant for the deeper understanding of the Hawking radiation and backreaction on it. Note that as pointed out above formally this effect appears due to the same exponential factor.

Now we present some intuitive speculations, which still need to get some solid mathematical approval. Let us see what happens with the light rays which are scattered off the surface of the star or radiated by it. If one has absolutely sensitive device and takes the above picture seriously, he is expecting to see that the signal will be eternally coming out from the star. Of cause sooner or letter the electromagnetic waves will be red–shifted to the radio frequency, but still the signal will be eternally coming out, although being exponential suppressed in frequency.

However, this picture is valid only if we assume that light rays are going along light–like geodesics on the Schwarzschild background. That is true, if one neglects that electromagnetic wave itself carries energy and, hence, also curves the space–time. The latter effect is very small, but the question is if one can neglect it, if address the issues of the exponentially suppressed effects.

To understand what we are actually up to here, consider the fig. (1) of the lecture IV. From the corresponding picture one can see that the horizon is just one of the light–like geodesics. It is a boundary between two families of "outgoing" geodesics. Moreover if light was going along the horizon it will remain there eternally.

However, if one takes into account that electromagnetic field also does curve space–time, he has to draw real light–like world–lines rather than geodesics. The picture of the fig. (1) from the lecture IV is not applicable anymore. In fact, apart from all the new picture will not be static. But certain relevant features of this picture will remain unchanged. Namely, we still expect to have two families of "outgoing" light–like world–lines — those which escape to infinity and those which are directed onto the singularity. To see this one can just study short parts of the world–lines in question, i.e. seeds of light–like lines during small periods of time. But the boundary separating these two families of curves will not belong to the class of the light–like world–lines. The light ray cannot anymore eternally stay on the fixed radius $r = r_g$.

Rephrasing this, we expect that among the photons, which are emitted by the star, there will be a last one that will reach outside observers. The next photon after that will just participate into the creation of the black hole as a part of its matter content. As the result, the outside observer sooner or latter will stop receiving signals from the collapsing star. And that will happen objectively rather than due to a lack of the sensitivity of his device.

One of the disadvantages of the picture that we have described here is that, if it is true, then the moment of the formation of the black hole depends on the energy of the photons that are emitted by the collapsing star. But at this moment for us it is important to see that the black hole is actually created during a finite time as measured by outside observers. In any case we just qualitatively described some phenomenon which remains to be described quantitatively somehow.

Problems

- Calculate the Christoffel symbols and the Ricci tensor for the metric ds^2_-.

Subjects for further study:

- Thin–shell collapse.
- Gluing conditions for the metric.
- Black hole thermodynamics.