Task 1: holomorphic functions, Cauchy formula, Taylor series. Deadline: February, 8

February 3, 2016

Problem 1. Find the $\mathbb{C}$-linear and $\mathbb{C}$-antilinear parts of the following $\mathbb{R}$- linear operators $L : \mathbb{C}^2 \to \mathbb{C}$, here $z = (z_1, z_2)$, $z_k = x_k + iy_k$:

a) $L(z) = x_1 + y_1$;

b) $L(z) = x_1 + y_2$;

c) $L(z) = x_1 + 2iy_2$;

d) $(1 + i)x_1 + iy_1 + 2x_2 + 3y_2$.

Problem 2. Are the following functions of two variables $f(z) = f(z_1, z_2)$ holomorphic at the origin?

a) $f(z) = x_1 + iy_2$;

b) $f(z) = x_1^2 + 2ix_1y_1 + y_1^2$;

c) $f(z) = x_1^2 + 2ix_1y_1 - y_1^2$;

d) $f(z) = \frac{z_1+z_2}{1+z_1}$;

e) $f(z) = \frac{z_1^2+z_2^3}{z_1^2+z_2^2}$;

f) $f(z) = \frac{z_1^4+z_2^4}{z_1^2+z_2^2}$.

Problem 3. Find which ones of the above functions are

a) continuous in a neighborhood of zero;

b) separately holomorphic (that is, holomorphic in each individual variable z_k) in a neighborhood of the origin.

Problem 4. Write the Taylor series for the following functions at the origin. Find their convergence domains and all the values of multiradii of convergence polydisks.

a) $f(z) = \frac{1}{1-z_1z_2^2}$;

b) $f(z) = \frac{1}{1-z_1-z_2^2}$;

c) $f(z) = \frac{1}{(1-z_1)(1+z_2)}$;

d) $f(z) = \frac{1}{(1-(z_1+z_2)^2)(1-z_2)}$;

e) $f(z) = \sin(z_1 + z_2^2)$.

Problem 5. Prove that the domain of convergence of any Taylor series is always *logarithmically convex:* if $r = (r_1, \dots, r_n)$, $\rho = (\rho_1, \dots, \rho_n)$ are two multiradii of convergence polydisks, then for every $\alpha \in [0, 1]$ the series converges at every point $z = (z_1, \dots, z_n)$ with $|z_j| < r_j^\alpha \rho_j^{1-\alpha}$.

Problem 6. Let $V \subset \mathbb{C}^n$ be a domain, $\Gamma \subset V$ be a one-dimensional complex submanifold. Let γ be a closed path in Γ contractible along Γ (i.e., contractible as a closed path in Γ).

a) Prove that the integral along γ of each holomorphic 1-form on V vanishes.

b) Is it true that the same integrals but along *every* closed path *contractible in* V vanish?

Problem 7. Calculate the following integrals:

a) $\oint_{\{|z|=1\}} \frac{\sin\zeta+1}{\zeta} d\zeta$, $z \in \mathbb{C}$;

b) $\oint_{\{|z_1|=1\}} \oint_{\{|z_2|=1\}} \frac{\zeta_1+\zeta_2}{\zeta_1-\frac{1}{2}} d\zeta_1 d\zeta_2;$

c) $\oint_{\{|z_1|=1\}} \oint_{\{|z_2|=1\}} \frac{\zeta_1+\zeta_2+1}{\zeta_1(\zeta_2-\frac{1}{2})};$

d) $\oint_{\{|z_1|=1\}} \oint_{\{|z_2|=\frac{1}{3}\}} \frac{\cos\zeta_1+\zeta_2}{\zeta_1(\zeta_2-\frac{1}{2})}.$