Task 3. Analytic sets, Proper Mapping Theorem, automorphisms. Deadline: March 9

February 22, 2016

Problem 1. * Let $P_w(z_1)$ be a Weierstrass polynomial on $\mathbb{C} \times \Delta$, Δ being a polydisk in $\mathbb{C}^{n-1}$. Let for every $w \in \Delta$ all its roots be contained in a given disk $D_\delta \subset \mathbb{C}$. Set

$$\Gamma = \{P_w(z_1) = 0\} \subset U = D_\delta \times \Delta; \ \pi(z_1, w) = w.$$

For every $w \in \Delta$ let $k(w)$ denote the number of geometrically distinct points of intersection $\pi^{-1}(w) \cap \Gamma$, $k_{\max} = \max_w k(w)$. Set

$$A = \{w \in \Delta \mid k(w) < k_{\max}\} \subset \Delta.$$

Prove that A is the zero locus of a holomorphic function $g : \Delta \to \mathbb{C}$:
a) in the case, when $P_w(z_1)$ is irreducible;
b) in the general case.

Problem 2. Prove **Weierstrass Division Theorem.** *Let $g(z_1, w)$ be a Weierstrass polynomial of degree d in z_1, $(z_1, w) \in \mathbb{C}^n$ (more precisely, its germ at $0 \in \mathbb{C}^n$). For every germ of holomorphic function $f(z_1, w)$ there exist a unique germ of holomorphic function h and a unique polynomial $p(z_1, w) = a_0(w)z_1^\nu + \cdots + a_\nu(w)$ in z_1, $\nu = deg_{z_1} p < d$, with coefficients $a_j(w)$ holomorphic at $w = 0$ such that*

$$f = hg + p. \tag{1}$$

a) Prove a version for one variable: for every germ of holomorphic function $F(u)$, $u \in \mathbb{C}$, at 0 and every $d \in \mathbb{N}$ there exist a unique germ of holomorphic function h and a unique polynomial $p(u)$ of degree less than d such that

$$F(0) = h(u)u^d + p(u).$$

b)* Let D_δ, Δ be as in the above problem (with $P_w(z_1)$ replaced by $g(z_1, w)$) such that f is holomorphic on $\overline{D}_\delta \times \Delta$. Prove formula (1) for

$$h(z_1, w) = \frac{1}{2\pi i} \oint_{|u|=\delta} \frac{f(u, w)}{g(u, w)} \frac{du}{u - z_1}.$$

Problem 3. Prove that the image of an irreducible analytic set under a proper holomorphic mapping is an irreducible analytic set. That is, consider an irreducible analytic subset $A \subset M$ in a complex manifold M and a holomorphic mapping $f : M \to N$ to a complex manifold N with proper restriction $f|_A : A \to N$. Then $f(A) \subset N$ is an irreducible analytic subset.
a) State and prove this statement for germs;
b) prove for global analytic sets.

Problem 4. Which ones of the following algebraic sets in $\mathbb{CP}^2$ or $\mathbb{CP}^3$ are regular?

a) $z_0^p + z_1^p + z_2^p = 0$, $p \geq 2$;

b) $z_0^2 z_1^2 + z_2^4 = 0$;

c) $z_0(z_1^2 + z_2^2) + z_3^3 = 0$;

d) $z_0^2(z_1^2 + z_2^2) + z_3^4 = 0$;

e) the closure in $\mathbb{CP}^3$ of the rational curve $t \mapsto (t, t^2, t^3) \in \mathbb{C}^3$, $t \in \mathbb{C}$;

f) the closure in $\mathbb{CP}^3$ of the rational curve $t \mapsto (t, t^2, t^4) \in \mathbb{C}^3$, $t \in \mathbb{C}$.

Problem 5. Find

a) which ones of the above algebraic sets are irreducible?

b) Present the reducible ones as unions of irreducible components.

Problem 6. Find a biholomorphic automorphism of the unit ball in $\mathbb{C}^2$ that sends the origin to the point

a) $(\frac{1}{2}, 0)$;

b) $(\frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}})$.

Problem 7. (a), b)). Find similar automorphisms of the unit polydisk in $\mathbb{C}^2$.

Problem 8. Does there exist a biholomorphic automorphism

a) of $\mathbb{CP}^1 = \overline{\mathbb{C}}$ transforming the points $0, 1, 2, \infty$ to $0, 1, 3, \infty$ respectively?

b) of $\mathbb{CP}^1 = \overline{\mathbb{C}}$ transforming the points $0, 1, \frac{1}{2}, 2$ to $0, 2, 1, 4$ respectively?

c) a non-trivial biholomorphic automorphism of $\mathbb{CP}^2$ fixing $(1:0:0)$, $(0:1:0)$, $(0:0:1)$ and a given $(a:b:c)$ with $a, b, c \neq 0$?

d) a non-trivial biholomorphic automorphism of $\mathbb{CP}^2$ fixing $(1:0:0)$, $(0:1:0)$, $(0:0:1)$ and a given point $(0:a:b)$?

Problem 9. Consider the action of the group of biholomorphic automorphisms of $\mathbb{CP}^2$ on quadruples of points $a = (A_1, A_2, A_3, A_4) \in (\mathbb{CP}^2)^4$.

a) Let two quadruples a, b be in generic position. How many are there biholomorphic automorphisms of $\mathbb{CP}^2$ transforming a to b?

b)* What could be the dimension of an orbit of the above action?

c)** Describe all the orbits.