Task 4. Automorphisms. Fatou–Bieberbach domains. Deadline: March 28

March 14, 2016

Problem 1. Find the lower term of k-th iterate of the germ of conformal mapping

$$f(z) = az + z^m + O(z^{m+1}),\ m \geq 2.$$

Problem 2. Consider a holomorphic mapping $f : U \to \mathbb{C}$, where $U \subset \mathbb{C}$ is a connected domain containing the origin. Let $f(0) = 0$ and $|f'(0)| = 1$. Prove that the germ of the mapping f at the origin is conformally conjugated to the linear germ $g(z) = \lambda z$, $\lambda = f'(0)$, if and only if f has a simply connected bounded invariant neighborhood $V \subset U$ of the origin. Treat separately the following cases:

a) λ is a root of unity.

b)* general case: $\lambda = e^{2\pi i\theta}$, $\theta \in \mathbb{R}$ is arbitrary.

Hint. Use Riemann Mapping Theorem and Schwarz Lemma.

Problem 3. Prove that for every polynomial $P(z)$, $degP \geq 2$, such that $P(0) = 0$ and

a) $P'(0) = -1$;

b) $P'(0) = e^{2\pi i \frac{p}{q}}$, $p, q \in \mathbb{Z}$

the germ of the mapping P at the origin is not conformally conjugated to its linear part $w \mapsto P'(0)w$.

Hint. Use a one-dimensional analogue of Cartan's Theorem.

Problem 4. * Provide an example of polynomial involution $f : \mathbb{C}^2 \to \mathbb{C}^2$, $f^2 = Id$ fixing the origin with $f'(0) = -Id$: this shows that the higher-dimensional analogues of the above statements are false.

Hint. Construct the above $f = (f_1, f_2)$ with $degf_j \leq 2$.

Problem 5. Let $U \subset \mathbb{C}^2$ be a Fatou-Bieberbach domain: the basin of a non-resonant attractive fixed point of a polynomial automorphism that does not coincide with all of $\mathbb{C}^2$. Prove that the intersection with U of every complex line is simply connected.

Hint. Use Maximum Principle.

Problem 6. Using the result of the above problem prove that the unit disk can be realized as a one-dimensional submanifold in $\mathbb{C}^2$.

Problem 7. Are the following germs at 0 of polynomial mappings linearizable by germ of biholomorphic coordinate change?

a) $F : (z_1, z_2) \mapsto (\frac{1}{2}z_1 + z_1^2, \frac{1}{4}z_2)$;

b) $F : (z_1, z_2) \mapsto (\frac{1}{3}z_1 + z_2^5, \frac{1}{2}z_2 + z_1^6)$;

c)* $F : (z_1, z_2) \mapsto (\frac{1}{4}z_1 + z_2^2, \frac{1}{2}z_2)$.

Problem 8. * Let $f = \frac{P}{Q} : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational mapping having an attracting fixing point at the origin: $f(0) = 0$, $\lambda = f'(0)$, $0 < |\lambda| < 1$. Set

$$U = \{z \in \overline{\mathbb{C}} \mid f^k(z) \to 0 \text{ as } k \to +\infty\}.$$

Let V denote the connected component of the open set U that contains the origin: this is the *immediate attractive basin.* Prove that the local linearizing conformal germ

$$h : (\mathbb{C}, 0) \to (\mathbb{C}, 0), \ \ h(0) = 0, \ \ h'(0) = 1, \ \ \lambda h = h \circ f$$

extends holomorphically to a epimorphic mapping $h : V \to \mathbb{C}$.