Task 5. Part 1: Holomorphic convexity. Deadline: April 20

April 6, 2016

Problem 1. Prove that every connected component of the interior of the intersection of
a) a finite number of holomorphically convex domains in $\mathbb{C}^n$,
b)* an infinite (may be uncountable) number of holomorphically convex domains in $\mathbb{C}^n$
is holomorphically convex.
Hint to b). Use Cartan–Thullen Theorem.

Definition. Let D be a domain in a complex manifold. Let $f_1, \dots, f_k : D \to \mathbb{C}$ be holomorphic functions such that the open subset

$$V = \{|f_j| < 1 \mid j = 1, \dots, k\} \subset D$$

has compact closure in D. The subset V is called an *analytic polyhedron.*

Problem 2. Prove that
a) each connected component of every analytic polyhedron is a holomorphically convex domain;
b) if $n < k$ and the mapping $f = (f_1, \dots, f_k) : V \to \mathbb{C}^k$ is injective and has rank n at each point, then $F(V)$ is a submanifold in $\Delta_{1,\dots,1}$: a so-called *Oka–Weil domain.*

Definition. A real C^∞ hypersurface S in a domain $D \subset \mathbb{C}^n$ (i.e., a submanifold of real codimension one) is *Levi-flat,* if for every $x \in S$ there exists a germ of holomorphic hypersurface $L_x \subset S$ through x, i.e., a germ of complex submanifold of complex codimension one in D that is contained in S.

Problem 3. Let $\phi : D \to \mathbb{R}$ be a C^∞ function with nonzero differential at each point, set

$$S = \{\phi = 0\}.$$

Prove that the hypersurface S is Levi-flat, if and only if the Levi form on TS defined by the function ϕ vanishes identically.

Definition. Let M be a k-dimensional manifold. Consider a C^∞ hyperplane field $H \subset TM$: $H_x \subset T_xM$ is a vector subspace of codimension one for every $x \in M$ that is C^∞ as a Grassmanian-valued function in x. Each $z \in M$ has a neighborhood $U = U(z) \subset M$ where there exists a differential 1-form ω, $\omega \neq 0$, such that $H = Ker(\omega)$. We say that H is a *contact structure,* if the 2-form $d\omega$ is non-degenerate on H_x for every $x \in M$.

Problem 4. a) Show that the above definition does not depend on the choice of the form ω;
b) Show that the contact structure may exist only in odd dimension k.
c) Let D, ϕ, S be the same, as in the previous problem (but now S is not Levi-flat). Let $H = T_{\mathbb{C}}S$ denote the field of the maximal complex subspaces $H_x = T_{x,\mathbb{C}}S \subset T_xS$, $x \in S$; this is a hyperplane field on S. Prove that H is a contact structure, if and only if the Levi form associated to the function ϕ is non-degenerate at each $x \in S$.
d) Prove that the latter property is invariant under biholomorphic transformations.

Problem 5. Let (Y, π) be the envelope of holomorphy of a Riemann domain. For a given $y \in Y$ let π_y^{-1} denote the germ of the inverse mapping π^{-1} at the point $z = \pi(y)$ such that $\pi_y^{-1}(z) = y$.

a) Prove that for every $y_1 \neq y_2 \in Y$ with $z = \pi(y_1) = \pi(y_2)$ there exists a holomorphic function $f : Y \to \mathbb{C}$ such that the germs of functions $(f \circ \pi_{y_1}^{-1}, z)$, $(f \circ \pi_{y_2}^{-1}, z)$ are distinct.

b)* Deduce from this that holomorphic functions on Y separate points: for every $x \neq y \in Y$ there exists a holomorphic function $f : Y \to \mathbb{C}$ such that $f(x) \neq f(y)$.