Task 5. Part 2: Dolbeault cohomology, Cousin problems, sheaves

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Problem 1. Let D be a topological space, $\mathcal{J}$ be a sheaf over it. Let $\mathcal{U}$, $\mathcal{V}$ be coverings of the space D by open sets such that $\mathcal{V} < \mathcal{U}$: each covering element $V_\alpha \in \mathcal{V}$ is contained in some element $U_{\phi(\alpha)} \in \mathcal{U}$ for a certain index mapping $\phi : \alpha \mapsto \phi(\alpha)$. For every two mappings ϕ_1, ϕ_2 as above let us consider the corresponding induced mappings

$$f_j : C^k_{\mathcal{U}} \to C^k_{\mathcal{V}}, \ (f_j h)_{\alpha_0 \dots \alpha_k} = h_{\phi_j(\alpha_0) \dots \phi_j(\alpha_k)}, \ j = 1, 2.$$

Show that f_1 and f_2 are chain-homotopic:

$$f_2 - f_1 = \delta\theta_k + \theta_{k+1}\delta, \ \delta : C^r \to C^{r+1} \text{ is the differential,}$$

$$\theta_l : C^l_{\mathcal{U}} \to C^{l-1}_{\mathcal{V}}, \ (\theta_l h)_{\alpha_0, \dots \alpha_{l-1}} = \sum_{s=0}^{l-1} (-1)^s h_{\phi_1(\alpha_0) \dots \phi_1(\alpha_s) \phi_2(\alpha_s) \dots \phi_2(\alpha_{l-1})}.$$

Problem 2. Let M be a real manifold. Let $H^k_{dR}(M)$ denote the de Rham cohomology: the quotient space of the C^∞-smooth closed k-forms over the exact k-forms (the image of the differential). Prove de Rham theorem: $H^k_{dR}(M)$ is isomorphic to the cohomology $H^k(M, \mathbb{R})$ with coefficients in the sheaf of locally constant real-valued functions. (Hence, it is isomorphic to the singular cohomology with real coefficients.)

Hint. Let $\mathcal{E}^k$ denote the sheaf of C^∞-smooth k-forms. Use the short exact sequences

$$0 \longrightarrow d\mathcal{E}^{k-1} \longrightarrow \mathcal{E}^k \longrightarrow d\mathcal{E}^k \longrightarrow 0, \ k \geq 0; \ d\mathcal{E}^{-1} := \mathbb{R},$$

the fact that the sections of the sheaf $d\mathcal{E}^{k-1}$ are the closed k-forms and triviality of the cohomology $H^s(M, \mathcal{E}^k)$ for $s > 0$ (as a given known statement).

Problem 3. Prove the above vanishing statement $H^s(M, \mathcal{E}^k) = 0$ using the same idea, as in the proof of the theorem from the lectures on the solution of Additive Smooth Cousin Problem, which implies that $H^1(M, C^\infty(M)) = 0$ (locally finite covering, partition of unity, etc.).

Problem 4. Prove that the $\bar{\partial}$-problem can be solved for punctured disk $D^0_1 = D_1 \setminus \{0\} \subset \mathbb{C}$.

Problem 5. Calculate the cohomology $H^k(D^0_1, \mathcal{O}^*)$, using the result of the latter problem, the known cohomology $H^k(D^0_1, \mathbb{Z})$ and appropriate exact sequences.

Problem 6. Let M be a compact complex manifold. Prove that the space of holomorphic sections of every holomorphic vector bundle on M is finite-dimensional.

Hint. Introduce a Banach space structure on the above space and prove compactness of its unit ball.