Several Complex Variables

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1 Holomorphic functions of several complex variables. Cauchy–Riemann equations, Cauchy formula, Taylor series

Definition 1.1 Let $\Omega \subset \mathbb{C}^n$ be an open subset. Recall that a function $f : \Omega \to \mathbb{C}$ is said to be ($\mathbb{R}$-)differentiable at a point $p \in \Omega$, if it is differentiable there as a function of real variables: there exists an $\mathbb{R}$-linear mapping $df(p) : T_p\mathbb{C}^n \simeq \mathbb{R}^{2n} \to T_p\mathbb{C} \simeq \mathbb{R}^2$ such that

$$f(z) - f(p) = df(p)(z-p) + o(z-p), \text{ as } z \to p.$$

A function f is said to be $\mathbb{C}$-*differentiable at a point* p, if it is differentiable there and its differential $df(p)$ is $\mathbb{C}$-linear. A function f is said to be *holomorphic* on Ω, if it is $\mathbb{C}$-differentiable at each point $x_0 \in \Omega$. A function f is said to be *holomorphic at a point* $x_0 \in \mathbb{C}^n$, if it is $\mathbb{C}$-differentiable in some its neighborhood. A holomorphic mapping $F = (F_1, \dots, F_m) : U \to V$, $U \subset \mathbb{C}^n$, $V \subset \mathbb{C}^m$ is defined in literally the same way: it is holomorphic, if and only if so are its components $F_1, \dots, F_m$.

Holomorphicity of a differentiable function is equivalent to Cauchy–Riemann Equations. To write them, let us recall the following definitions and formula from one-dimensional complex analysis. Let $f : U \to \mathbb{C}$ be a differentiable mapping of a domain $U \subset \mathbb{C}$. The differential $df(p) : T_p\mathbb{C} \simeq \mathbb{C} \to T_{f(p)}\mathbb{C} \simeq \mathbb{C}$ is an $\mathbb{R}$-linear map $\mathbb{C} \to \mathbb{C}$. Each $\mathbb{R}$-linear map $L : \mathbb{C} \to \mathbb{C}$ is a sum of its $\mathbb{C}$-linear part and its $\mathbb{C}$-antilinear part:

$$L = Adz + B\overline{dz}; \ A, B \in \mathbb{C}.$$

The coefficient A of the $\mathbb{C}$-linear part of the differential is called $\frac{\partial f}{\partial z}(p)$; the coefficient B of the antilinear part is called $\frac{\partial f}{\partial \bar{z}}(p)$. Let $z = x + iy$. One has

$$df = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar{z}}\overline{dz}; \ \frac{\partial f}{\partial z} = \frac{1}{2}(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}), \ \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}).$$

Proposition 1.2 *(Cauchy–Riemann Equations). A differentiable function $f(z_1, \dots, z_n)$ on a domain in $\mathbb{C}^n$ is holomorphic, if and only if*

$$\frac{\partial f}{\partial \bar{z}_j} \equiv 0 \text{ for every } j = 1, \dots, n. \qquad (1.1)$$

Proof The tangent space $T_p\mathbb{C}^n$ is the direct sum of complex "coordinate lines" parallel to the coordinate axes. Thus, the $\mathbb{C}$-linearity of the differential

$df(p)$ is equivalent to the $\mathbb{C}$-linearity of its restrictions to all the coordinate lines. The latter is equivalent to (1.1). This proves the proposition. $\square$

Example 1.3 Holomorphicity is preserved under arithmetic combinations and compositions. In particular, polynomials and rational functions and in general, all the elementary functions (restricted to their appropriate definition domains) are holomorphic.

Remark 1.4 In the case, when $n = 1$ the above definition coincides with the classical definition of holomorphic function of one complex variable. If a function f is holomorphic in Ω, then for every complex line $L \subset \mathbb{C}^n$ the restriction $f|_{L\cap\Omega}$ is holomorphic as a function of one variable. The next Big Hartogs' Theorem implies that the converse is also true.

Theorem 1.5 *(Hartogs). A function $f(z_1, \dots, z_n)$ is holomorphic on a domain $\Omega = \Omega_1 \times \dots \times \Omega_n \subset \mathbb{C}^n$, if and only if it is* **separately holomorphic:** *for every $j = 1, \dots, n$ and every given collection of points $z_s \in \Omega_s$, $s \neq j$, the function $g(z) = f(z_1, \dots, z_{j-1}, z, z_{j+1}, \dots, z_n)$ is holomorphic on Ω_j.*

Remark 1.6 The nontrivial part of the theorem says that if a function is separately holomorphic, then it is holomorphic as a function of several variables. Under the additional assumption that f is differentiable, this statement follows immediately from Proposition 1.2. We will not prove Theorem 1.5 in full generality. We will prove its weaker version under continuity assumption (Osgood Lemma).

Holomorphic functions in several variables share the basic properties of holomorphic functions in one variable: existence of converging Taylor series, uniqueness of analytic extension, openness, Maximum Principle, Liouville Theorem. At the same time we will see that the following new phenomena hold for holomorphic functions in several complex variables, which are in contrast with the case of one variable:

- *no isolated singularities;*

- *erasing compact singularities:* holomorphic functions on a complement of a domain $V \subset \mathbb{C}^n$ to a compact subset $K \Subset V$ extend holomorphically to all of V.

Everywhere below for every $\delta > 0$ and $z \in \mathbb{C}$ we denote

$$D_\delta(z) = \{|w - z| < \delta\} \subset \mathbb{C};\ D_\delta = D_\delta(0).$$

The corresponding balls in $\mathbb{C}^n$ of radius δ will be denoted by $B_\delta(z)$ and B_δ respectively. For every $r = (r_1, \dots, r_n) \in \mathbb{R}_+^n$, $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ the *polydisk* of multiradius r centered at z is the product of disks of radii r_j, which we will denote by

$$\Delta_r(z) = \prod_j D_{r_j}(z_j) = \{w = (w_1, \dots, w_n) \in \mathbb{C}^n \mid |w_j - z_j| < r_j\}; \ \Delta_r = \Delta_r(0).$$

For $\delta > 0$ we denote $\Delta_\delta(z) = \Delta_{(\delta,\dots,\delta)}(z)$, $\Delta_\delta = \Delta_\delta(0)$. In the case, when we would like to specify the dimension of the ambient space of the polydisk, we will write Δ_r^n, $\Delta_\delta^n(z)$ etc.

The next theorem generalizes Cauchy formula for holomorphic functions in one variable.

Theorem 1.7 *(Multidimensional Cauchy formula). Let $f : \overline{\Delta}_r \to \mathbb{C}$ be a continuous function that is holomorphic in each variable z_j, $j = 1, \dots, n$. (In particular, this holds for every function holomorphic on Δ_r and continuous on its closure). Then for every $z = (z_1, \dots, z_n) \in \Delta_r$ one has*

$$f(z) = \frac{1}{(2\pi i)^n} \oint_{|\zeta_1|=r_1} \dots \oint_{|\zeta_n|=r_n} \frac{f(\zeta)}{\prod_{j=1}^n (\zeta_j - z_j)} d\zeta_n \dots d\zeta_1. \tag{1.2}$$

Remark 1.8 Let $g(\zeta)$ denote the sub-integral function in the latter right-hand side. The multiple integral in (1.2) is independent of integration order (Fubini's theorem and continuity of the function $g(\zeta)$). It is equal to the integral of the complex-valued differential n-form $g(\zeta)d\zeta_1 \wedge \dots \wedge d\zeta_n$ on the n-torus $\mathbb{T}^n = \prod_{j=1}^n S_j^1$, $S_j^1 = \{|\zeta_j| = r_j\}$, oriented as a product of positively (i.e., counterclockwise) oriented circles. That is, an orienting basis $v_1, \dots, v_n \in T_\zeta \mathbb{T}^n$ is formed by vectors $v_j \in T_{\zeta_j} S_j^1$ oriented counterclockwise.

Proof It suffices to prove the statement of the theorem in the case, when f is holomorphic in each variable on a domain containing the closed polydisk $\overline{\Delta}_r$: the general case is reduced to it via scaling the function f to $f_\varepsilon(z) = f(\varepsilon z)$, $0 < \varepsilon < 1$ (which is holomorphic in each variable on $\overline{\Delta}_r$) and passing to the limit under the integral, as $\varepsilon \to 1$. We prove formula (1.2) by induction in n.

Induction base: for $n = 1$ this is the classical Cauchy formula for one variable.

Induction step. Let formula (1.2) be proved for the given $n = k$. Let us prove it for $n = k + 1$. For every $w = (w_1, \dots, w_k) \in \mathbb{C}^k$ set

$$f_w(t) = f(w_1, \dots, w_k, t).$$

For every fixed $z_{k+1} \in D_{r_{k+1}}$ the function $g(w_1, \dots, w_k) = f_w(z_{k+1})$ is holomorphic on $\overline{\Delta}_{(r_1,\dots,r_k)}$. Hence,

$$f(z_1, \dots, z_{k+1}) = \frac{1}{(2\pi i)^k} \oint_{|\zeta_1|=r_1} \dots \oint_{|\zeta_k|=r_k} \frac{f_\zeta(z_{k+1})}{\prod_{j=1}^k (\zeta_j - z_j)} d\zeta_k \dots d\zeta_1, \quad (1.3)$$

by the induction hypothesis. The function $f_\zeta(t)$ being holomorphic in $t \in \overline{D}_{r_{k+1}}$ for every $\zeta = (\zeta_1, \dots, \zeta_k)$, it is expressed by Cauchy Formula

$$f_\zeta(t) = \frac{1}{2\pi i} \oint_{|\zeta_{k+1}|=r_{k+1}} \frac{f_\zeta(\zeta_{k+1})}{\zeta_{k+1} - t} d\zeta_{k+1} \text{ for every } t \in D_{r_{k+1}}.$$

Substituting the latter formula with $t = z_{k+1}$ to (1.3) yields (1.2), by continuity and Fubini Theorem. □

Lemma 1.9 *(Osgood). Every continuous function on a domain in $\mathbb{C}^n$ that is holomorphic in each individual variable is holomorphic.*

Proof It sufficed to prove the statement of the lemma for a function continuous on a closed polydisk $\overline{\Delta}_r$. Then Multidimensional Cauchy Formula (1.2) holds, and its subintegral expression is a continuous family of rational functions in $z \in \Delta_r$. Therefore, the subintegral expressions are holomorphic on Δ_r. They are uniformly bounded and continuous together with derivatives on compact subsets in Δ_r. Therefore, the integral is C^1-smooth and its partial derivatives are equal to the integrals of partial derivatives in z of the subintegral expression (here one can differentiate the integral by the above boundedness and continuity statements). It satisfies Cauchy–Riemann equations, as do the subintegral functions, and hence, is holomorphic. The lemma is proved. □

Set

$$\mathbb{Z}_{\geq 0} = \mathbb{N} \cup \{0\}.$$

Theorem 1.10 *Every function f holomorphic at $0 \in \mathbb{C}^n$ is a sum of power series converging to f uniformly on a neighborhood of 0:*

$$f(z) = \sum_{k \in \mathbb{Z}^n_{\geq 0}} c_k z^k; \;\; c_k \in \mathbb{C}, \;\; z^k = z_1^{k_1} \dots z_n^{k_n}, \;\; c_0 = f(0). \quad (1.4)$$

Proof Fix a $\delta > 0$ such that f is holomorphic on the closed polydisk $\overline{\Delta}_\delta$. Let us show that the right-hand side of the Cauchy formula is a sum of power series converging on Δ_δ. For every ζ_j and z_j with $|z_j| < \delta = |\zeta_j|$ one has

$$\frac{1}{\zeta_j - z_j} = \zeta_j^{-1}\frac{1}{1-\frac{z_j}{\zeta_j}} = \sum_{l=0}^{+\infty} \zeta_j^{-l-1} z_j^l. \tag{1.5}$$

This series converges absolutely uniformly on every disk $|z_j| \leq \delta'$ with $\delta' < \delta$. Hence, the product of the latter series for all $j = 1, \dots, n$ also absolutely uniformly converges to $\frac{1}{\prod_j(\zeta_j - z_j)}$ on $\Delta_{\delta'}$. Substituting formulas (1.5) for all j to (1.2) together with permutability of integration and series summation (ensured by absolute uniform convergence of subintegral series and uniform boundedness of the function on $\partial\Delta$) yields (1.4) with

$$c_k = \frac{1}{(2\pi i)^n}\oint_{|\zeta_1|=\delta'} \dots \oint_{|\zeta_n|=\delta'} \frac{f(\zeta)}{\zeta_1^{-k_1-1}\dots\zeta_n^{-k_n-1}} d\zeta_1 \dots d\zeta_n. \tag{1.6}$$

Substituting $k = 0$ yields $c_0 = f(0)$, by (1.2). □

2 Convergence of power series and convergence radius. Equivalent definition of holomorphic function

Lemma 2.1 (Abel). *Consider a power series $\sum_{k\in\mathbb{Z}^n_{\geq 0}} c_k z^k$. Let its terms $c_k z^k$ at a given point $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ be uniformly bounded, set $r_j = |z_j|$, $r = (r_1, \dots, r_n)$. Then the series converges uniformly on compact subsets in the polydisk Δ_r.*

Proof Fix some $\delta = (\delta_1, \dots, \delta_n)$ with $\delta_j < r_j$ for all j. It suffices to show that $\sum |c_k|\delta^k < \infty$. Indeed, set

$$\nu_j = \frac{\delta_j}{r_j} < 1, \ C = \sup_k |c_k r^k| < +\infty.$$

Then $|c_k|\delta^k \leq C\nu^k$. But

$$\sum_k \nu^k = \prod_{j=1}^n \left(\sum_{s=0}^{+\infty} \nu_j^s\right) = \frac{1}{\prod_j(1-\nu_j)} < +\infty.$$

Therefore, the series $\sum_k |c_k|\delta^k$ is majorated by a converging series $C\sum_k \nu^k$, and hence, converges. The lemma is proved. □

Definition 2.2 A polydisk $\Delta_r(a)$ with multiradius $r = (r_1, \dots, r_n)$ is called the *convergence polydisk* of a series $\sum_{k \in \mathbb{Z}_{\geq 0}} c_k (z-a)^k$, if the series converges in $\Delta_r(a)$ and does not converge in every polydisk $\Delta_R(a)$, with $R_j \geq r_j$ for all j and $R_j > r_j$ for at least some j. The multiradius of a convergence polydisk is called a *convergence multiradius.* (In general, the convergence multiradius is not unique, as we will see in the next examples.)

Definition 2.3 The *convergence domain* of a power series is the interior of the set of convergence points.

Corollary 2.4 *The convergence domain is a union of convergence polydisks. If $a = 0$, then the convergence domain is invariant under the torus action $(z_1, \dots, z_n) \mapsto (e^{i\phi_1} z_1, \dots, e^{i\phi_n} z_n)$, $\phi_j \in \mathbb{R}(mod 2\pi\mathbb{Z})$.*

Proof It suffices to prove the statements of the corollary for $a = 0$. Given a power series, let $\mathcal{D}$ denote its convergence domain. Given a point $z = (z_1, \dots, z_n) \in \mathcal{D}$, let us construct a polydisk $\Delta_r \subset \mathcal{D}$ containing z. For every $\lambda > 1$ close enough to 1 (dependently on z) one has $w = \lambda z \in \mathcal{D}$, by definition. Set $r_j = |w_j| = \lambda |z_j| > |z_j|$, $r = (r_1, \dots, r_n)$. The sequence $c_k r^k$ is uniformly bounded, by the convergence of the series $\sum_k c_k w^k$. Therefore, $\Delta_r \subset \mathcal{D}$ (Abel's Lemma 2.1) and $z \in \Delta_r$, by construction. The first statement of the corollary is proved. Its second statement follows from the first one and the invariance of each polydisk centered at 0 under the torus action. The corollary is proved. □

Remark 2.5 Each power series converges uniformly on compact subsets in its convergence domain, by Abel's Lemma 2.1 and the above corollary.

Example 2.6 The convergence domain of the series $\sum_{k \geq 0} z_1^k$ in two variables (z_1, z_2) is the cylinder $|z_1| < 1$. The latter cylinder is the unique convergence bidisk $\Delta_{1,\infty}$. The convergence domain of the series $\sum z_1^{k_1} z_2^{k_2}$ is the unit bidisk $\Delta_{1,1}$, which is the unique convergence bidisk. The convergence domain of the series $\sum (z_1 z_2)^k$ is the set $\{|z_1 z_2| < 1\}$, and every $r = (r_1, r_2)$ with $r_1 r_2 = 1$, $r_1, r_2 > 0$ is a convergence multiradius.

Let us recall that the convergence radius r of a power series $\sum_k c_k z^k$ in one variable is given by the classical Cauchy-Hadamard formula $r = (\overline{\lim}_{k\to\infty} c_k^{\frac{1}{k}})^{-1}$, or equivalently,

$$\overline{\lim}_{k\to\infty} (c_k r^k)^{\frac{1}{k}} = 1.$$

The next proposition generalizes this formula to several variables.

Proposition 2.7 *An $r = (r_1, \dots, r_n)$ with $r_j > 0$ is the multiradius of a convergence polydisk of a given power series $\sum c_k z^k$, if and only if*

$$\phi(r) = \overline{\lim}_{k\to\infty}(|c_k|r^k)^{\frac{1}{|k|}} = 1, \;\; r^k = r_1^{k_1} \dots r_n^{k_n}. \tag{2.1}$$

In the proof of the proposition we use homogeneity of the upper limit ϕ:

$$\phi(\lambda r) = \lambda\phi(r) \text{ for every } \lambda > 0.$$

First let us prove the following claim.

Claim. *A polydisk Δ_r is contained in the convergence domain $\mathcal{D}$ of a series $\sum_k c_k z^k$, if and only if $\phi(r) \leq 1$.*

Proof If $\Delta_r \subset \mathcal{D}$, then the series converges uniformly on compact subsets in Δ_r, hence, $c_k z^k \to 0$, as $k \to \infty$, for every z with $|z_j| < r_j$, in particular, for $z = \lambda r$ with arbitrary $0 < \lambda < 1$. Therefore, for every $0 < \lambda < 1$ the sequence $\lambda^{|k|}|c_k|r^k$ is uniformly bounded, and hence, the upper limit of the $|k|$-th roots of its terms is no greater than 1. Thus, $\phi(r) \leq \lambda^{-1}$ for every $\lambda \in (0, 1)$, hence, $\phi(r) \leq 1$. Conversely, let $\phi(r) \leq 1$. Then for every $\lambda > 1$ and every k large enough dependently on λ one has $|c_k|r^k < \lambda^{|k|}$, thus, $|c_k|(\frac{r}{\lambda})^k < 1$. Therefore, $\Delta_{r/\lambda} \subset \mathcal{D}$ for every $\lambda > 1$ (Abel's Lemma 2.1), and hence, $\Delta_r \subset \mathcal{D}$. The claim is proved. □

Proof of Proposition 2.7. Let r be a convergence multiradius. Then $\phi(r) \leq 1$, by the claim. If $\phi(r) < 1$, then $\phi(\lambda r) < 1$ for some $\lambda > 1$, and hence, $\Delta_{\lambda r} \subset \mathcal{D}$, by the claim. Thus, the convergence domain contains a bigger polydisk $\Delta_{\lambda r} \Supset \Delta_r$, and r is not a convergence multiradius. The contradiction thus obtained proves the proposition. □

Now let us prove that every holomorphic function is C^∞-smooth using the fact that it is locally the sum of a converging power series. We show that the latter is its Taylor series.

The higher derivatives $\frac{\partial^l f}{\partial z^l}$, $\frac{\partial^{k+l} f}{\partial z^k \partial \bar{z}^l}$ of function of one variable and the higher derivatives

$$\frac{\partial^{k+l} f}{\partial z^k \partial \bar{z}^l} = \frac{\partial^{k+l} f}{\partial z_1^{k_1} \dots \partial z_n^{k_n} \partial \bar{z}_1^{l_1} \dots \partial \bar{z}_n^{l_n}}, \;\; k, l \in \mathbb{Z}_{\geq 0}$$

of a function of n complex variables are defined by subsequent differentiations. They are independent on the choice of order of differentiations (if the order of smoothness of the function is no less than the number of differentiations). This follows from the general fact that every two differential operators with constant coefficients commute.

Example 2.8 Let $f(z) = z_1^{s_1} \dots z_n^{s_n}$. Then

$$\frac{\partial^{k+l} f}{\partial z_1^{k_1} \dots \partial z_n^{k_n} \partial \bar{z}_1^{l_1} \dots \partial \bar{z}_n^{l_n}} = 0 \text{ whenever } l \neq 0;$$

$$\frac{\partial^k f}{\partial z^k} = 0 \text{ whenever } k_j > s_j \text{ for a certain } j;$$

$$\frac{\partial^k f}{\partial z^k} = \prod_{j=1}^{n} \frac{s_j!}{(s_j - k_j)!} z^{s-k}, \text{ whenever } k_j \leq s_j \text{ for all } j.$$

Remark 2.9 All the above statements on power series remain valid for power series $\sum_k c_k (z-p)^k$ with arbitrary $p \in \mathbb{C}^n$: the convergence domain is a union of polydisks centered at p, etc.

Proposition 2.10 *Let a power series $f(z) = \sum_k c_k z^k$ has a non-empty convergence domain. Then its sum $f(z)$ is holomorphic and C^∞-smooth there and*

$$c_0 = f(0), \ c_k = \frac{1}{k_1! \dots k_n!} \frac{\partial^{|k|} f}{\partial z^k}(p), \tag{2.2}$$

Proof Without loss of generality we consider that $p = 0$. The convergence domain is a union of convergence polydisks. Fix a convergence polydisk Δ_r and let us prove the above regularity statements in Δ_r. We claim that each derivative (of any order) of the series $\sum_k c_k z^k$ converges uniformly on compact subsets in Δ_r. Let $\phi(r)$, $\phi_1(r)$ denote respectively the upper limits (2.1) corresponding to the initial series and its derivative

$$\frac{\partial}{\partial z_1}(\sum_k c_k z^k) = \sum_k k_1 z_1^{-1} c_k z^k.$$

One has

$$\phi_1(r) = \overline{\lim}_{k\to\infty}((|k_1 r_1^{-1} c_k| r^k)^{\frac{1}{|k|-1}} \leq \overline{\lim}_{k\to\infty}(|c_k| r^k)^{\frac{1}{|k|-1}} = \phi(r) \leq 1.$$

Thus, the above derivative series converge uniformly on compact subsets in Δ_r, by Proposition 2.7. For higher derivatives the proof is analogous: the l-th derivation yields a new multiplier polynomial in k of fixed degree $|l|$, and its contribution to the above upper limit cancels out after taking a root of order $|k|$, as in the above inequality. This implies infinite differentiability of the function f, and each its partial derivative is equal to the sum of the corresponding derivative series. In particular, $\frac{\partial f}{\partial \bar{z}_j} = 0$, since this holds for each term of the power series. Hence, f is holomorphic. The value $\frac{\partial^{|k|} f}{\partial z^k}(0)$ is equal to the free term of the corresponding derivative series, i.e., $k_1! \dots k_n! c_k$. This proves (2.2) and the proposition. $\square$

Corollary 2.11 *A function f on a domain $V \subset \mathbb{C}^n$ is holomorphic, if and only if each point $p \in V$ has a neighborhood where f is a sum of a converging power series $\sum_k c_k (z-p)^k$. The coefficients c_k are given by formula (2.2). Each holomorphic function is C^∞-smooth.*

The corollary follows from the above proposition and Theorem 1.10.

3 Analytic extension. Erasing singularities. Hartogs Theorem

Theorem 3.1 *(Uniqueness of analytic extension). Every two holomorphic functions on a connected domain $\Omega \subset \mathbb{C}^n$ that are equal on an open subset coincide on all of Ω.*

Proof It is sufficient to show that if a holomorphic function f on a connected domain Ω vanishes on some open subset $V \subset \Omega$, then $f \equiv 0$ on all of Ω. To do this, let us consider the subset

$$K = \cap_{k \in (\mathbb{Z}_{\geq 0})^n} \{ \frac{\partial^{|k|} f}{\partial z^k} = 0 \} \subset \Omega : \; K \supset V.$$

One has $f|_K \equiv 0$, since the latter intersection includes $k = 0$. The subset $K \subset \Omega$ is closed, being an infinite intersection of closed subsets, since $f \in C^\infty(\Omega)$ (Corollary 2.11). The set K is open. Indeed, at each point $p \in K$ the function f has vanishing Taylor series coefficients, by definition and formula (2.2). Hence, $f \equiv 0$ on a neighborhood of the point p, and thus, the latter neighborhood is contained in K. Therefore, K is a nonempty closed and open subset of a connected domain Ω, hence $K = \Omega$ and $f \equiv 0$ on Ω. $\square$

Proposition 3.2 (Openness Principle.) *Each non-constant holomorphic function on a connected domain is an open map: the image of each open subset is open.*

Proof Let f be a non-constant holomorphic function on a connected domain Ω. It suffices to show that for every point $z \in \Omega$ the image of arbitrary ball centered at z contains a neighborhood of the image $f(z)$. Fix a $z \in \Omega$

and a complex line L through z where $f|_L \not\equiv const$ in a neighborhood of z. The line L exists since f is locally non-constant (uniqueness of analytic extension). The restriction of the function f to a disk in $L \cap \Omega$ centered at z is an open map, being a non-constant holomorphic function of one complex variable. This implies that the image of every disk as above contains a neighborhood of the point $f(z)$, and hence, so does the image of arbitrary ball in Ω centered at z. The proposition is proved. $\square$

Corollary 3.3 (Maximum Principle.) *The module of a non-constant holomorphic function on a connected domain Ω cannot achieve its maximum in Ω.*

Proof If a module of a holomorphic function $f \not\equiv const$ achieves its maximum at a point $z \in \Omega$, then the image $f(\Omega)$ contains the point $f(z)$ but avoids the exterior of the circle through $f(z)$ centered at 0. Hence, it contains no its neighborhood, – a contradiction to Openness Principle. The corollary is proved. $\square$

Theorem 3.4 (Liouville). *Every bounded holomorphic function on all of $\mathbb{C}^n$ is constant.*

Proof The restriction of a bounded holomorphic function f to each complex line through the origin is constant, being a bounded holomorphic function on $\mathbb{C}$ (Liouville Theorem in one variable). Therefore, $f \equiv f(0)$ on $\mathbb{C}^n$. $\square$

It is known that for every domain $V \subset \mathbb{C}$ there exists a holomorphic function on V that extends analytically to no point of its boundary. This statement is false in higher dimensions. A basic counterexample, the *Hartogs Figure* is provided by the next theorem.

Theorem 3.5 *(Hartogs) Let $R = (R_1, \dots, R_n)$, $R_j > 0$, $1 \leq k < n$, $r = (r_1, \dots, r_k)$, $r_s < R_s$. Set $R^k = (R_1, \dots, R_k)$, $R^{n-k} = (R_{k+1}, \dots, R_n)$. Let $V \subset \Delta_{R^{n-k}} \subset \mathbb{C}^{n-k}$ be an open subset. Let $z = (z_1, \dots, z_n)$ be coordinates on $\mathbb{C}^n$. Set $t = (z_1, \dots, z_k)$, $w = (z_{k+1}, \dots, z_n)$,*

$$A = (\Delta_{R^k} \setminus \overline{\Delta_r}) \times \Delta_{R^{n-k}}, \ B = \Delta_{R^k} \times V \subset \Delta_R \subset \mathbb{C}^n, \ \Omega = A \cup B.$$

(In the case, when $n = 2$, $k = 1$, $V = D_{r_2}$, $r_2 < R_2$, the domain Ω is the so-called **Hartogs Figure,** *see Fig.1.) Then every function holomorphic on Ω extends holomorphically to the whole polydisk $\Delta_R = \Delta_{R^k} \times \Delta_{R^{n-k}}$.*

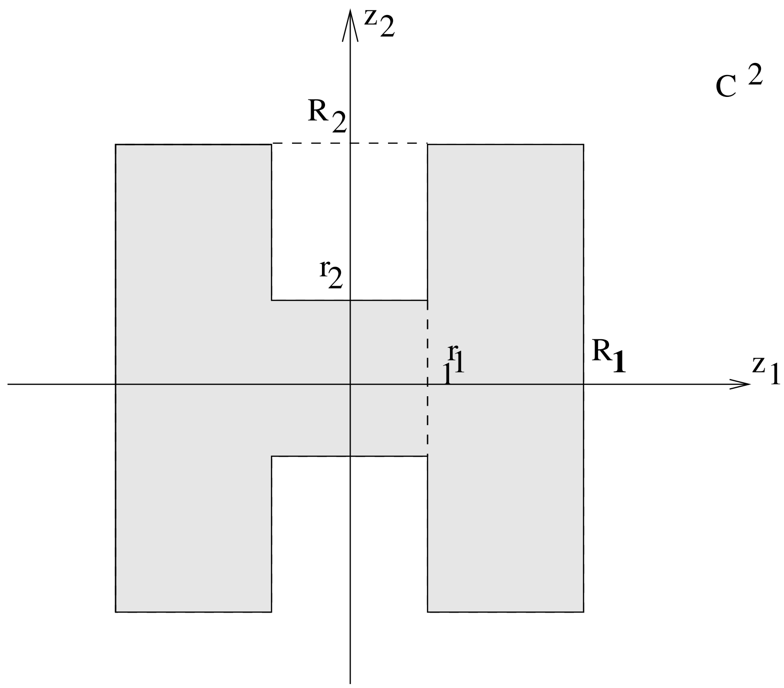

Figure 1: The Hartogs Figure.

Proof For simplicity, let us prove the theorem in the case, when $n = 2$, $k = 1$: thus $R^k = R_1$, $R^{n-k} = R_2$, $z = (z_1, z_2)$, $t = z_1$, $w = z_2$. The proof in the general case is analogous. Let f be a function holomorphic on Ω. Fix an arbitrary $\delta \in (r_1, R_1)$. For every $w \in V$ the function $f(z_1, w)$ is holomorphic in $z_1 \in D_{R_1} \subset \mathbb{C}$, since $D_{R_1} \times \{w\} \subset B \subset \Omega$. Therefore, for every $z_1 \in D_\delta$ it is expressed as Cauchy integral

$$f(z_1, w) = \frac{1}{2\pi i} \oint_{|z_1|=\delta} \frac{f(\zeta, w)}{\zeta - z_1} d\zeta. \tag{3.1}$$

For every fixed $w \in D_{R_2}$ the subintegral function is holomorphic in $z_1 \in D_\delta$. Hence, the integral is also holomorphic in $z_1 \in D_\delta$, as in the proof of Osgood's Lemma. For every fixed $\zeta \in D_{R_1} \setminus D_\delta \supset \partial D_\delta$ the function $f(\zeta, w)$ is holomorphic in $w \in D_{R_2}$, since $\{\zeta\} \times D_{R_2} \subset A \subset \Omega$. Finally, the subintegral function is holomorphic in $(z_1, w) \in D_\delta \times D_{R_2}$, and hence, so is the integral. Thus, formula (3.1) extends the function $f(z_1, w)$ holomorphically to $D_\delta \times D_{R_2}$. Thus, f is holomorphic there and hence, on all of $\Delta_R = D_{R_1} \times D_{R_2}$, since δ is an arbitrary number between r_1 and R_1. This proves the theorem for $n = 2$ and $k = 1$. Theorem 3.5 is proved. $\square$

Exercise. Prove Theorem 3.5 in the general case using multidimensional Cauchy integral.

Theorem 3.6 *(Erasing compact singularities). Let $G \subset \mathbb{C}^n$ be an open subset, $K \Subset G$ be a compact subset. Let both G and the complement $G \setminus K$ be connected. Then every function holomorphic on $G \setminus K$ extends holomorphically to all of G.*

We prove this theorem only in the case, when the ambient domain is a polydisk. Its proof in general case is more complicated and can be done by using, e.g., Bochner–Martinelli integral formula.
Proof of Theorem 3.6 in the case, when G is a polydisk. Let us prove the theorem in the case when $n = 2$: in higher dimensions the proof is literally analogous. Let $G = \Delta_R$, $R = (R_1, R_2)$. Let K_1, K_2 denote respectively the images of the compact set K under the projections to the z_1- and z_2-axes: $K_1 \Subset D_{R_1}$, $K_2 \Subset D_{R_2}$. Fix an open subset $V \subset D_{R_2} \setminus K_2$ and a $0 < r_1 < R_1$ such that $K_1 \Subset D_{r_1}$. Let Ω be the Hartogs figure from Theorem 3.5 constructed by the chosen r_1, V and R. One has $\Omega \subset \Delta_R \setminus K$. Therefore, every function holomorphic on $\Delta_R \setminus K$ is holomorphic on Ω, and hence, extends to a function holomorphic on all of Δ_R, by Theorem 3.5. $\square$

The next theorems concern holomorphic functions on complex manifolds (i.e., functions holomorphic in each holomorphic chart).

Theorem 3.7 *Let f be a bounded function holomorphic on the complement $\Omega \setminus \{g = 0\}$ of a complex manifold Ω (e.g., $\Omega \subset \mathbb{C}^n$) to the zero set of a locally non-constant holomorphic function $g : \Omega \to \mathbb{C}$. Then f extends holomorphically to all of Ω.*

Proof Without loss of generality we consider that $n = 2$: the proof in higher dimension is literally analogous. Fix a point $p \in A = \{g = 0\}$. Let us show that f limits continuously to a neighborhood of the point p in A and the limit is holomorphic. Fix a local holomorphic chart covering a neighborhood of the point p, which is identified with a domain in $\mathbb{C}^2$. Fix a complex line L through p where $g|_L \not\equiv 0$. Let us choose the coordinates (z_1, z_2) in the above chart so that $p = 0$, L is the z_1-axis. Then $g(z_1, 0)$ is a non-constant holomorphic function in one variable, and the origin is its isolated zero. Fix a circle $C = \{|z_1| = \sigma\} \subset \mathbb{C}$ centered at 0 where $g(z_1, 0) \neq 0$. Fix a $\delta > 0$ such that $g \neq 0$ on $C \times D_\delta$. The function $f(z_1, 0)$ is holomorphic and bounded on a punctured neighborhood of zero, and hence, extends holomorphically there (theorem on erasing isolated singularities of bounded holomorphic functions in one variable). Similarly, for every $z_2 \in D_\delta$ the function $g(z_1, z_2)$ in z_1 is non-constant and thus, has isolated zeros. The latter are again removable singularities for f. Thus, $f(z_1, z_2)$ is holomorphic in $z_1 \in D_\sigma$ for every fixed $z_2 \in D_\delta$. Therefore,

$$f(z_1, z_2) = \frac{1}{2\pi i} \oint_C \frac{f(\zeta, z_2)}{\zeta - z_1} d\zeta \text{ for every } (z_1, z_2) \in \Delta_{\sigma,\delta} = D_\sigma \times D_\delta.$$

The latter subintegral expression is holomorphic in the bidisk $\Delta_{\sigma,\delta}$ and coincides with $f(z_1, z_2)$ in the complement of the set A. This yields the desired analytic extension of the function f to a neighborhood of arbitrary point $p \in A$ and proves the theorem. $\square$

Theorem 3.8 *Let Ω be a complex manifold, and let $S \subset \Omega$ be a complex submanifold of codimension at least two. Then every function holomorphic on $\Omega \setminus S$ extends holomorphically to all of Ω.*

Proof Let $k \geq 2$ be the codimension of the submanifold S. Fix an arbitrary point $p \in S$ and a holomorphic chart $(z_1, \dots, z_n)$ identifying its neighborhood U with a polydisk $\Delta_R \subset \mathbb{C}^n$ so that p corresponds to the origin and the intersection $S \cap U$ corresponds to the $(n-k)$-coordinate subspace $(z_{k+1}, \dots, z_n)$, more precisely, the $(n-k)$-slice $\{0\} \times \Delta_{R^{n-k}} \subset \Delta_R$, $R^{n-k} = (R_{k+1}, \dots, R_n)$. From now on we denote by S the latter slice. Let f be a function holomorphic on $\Delta_R \setminus S$. Let us show that f extends holomorphically to all of Δ_R. This will imply the statement of the theorem. Set $t = (z_1, \dots, z_k)$, $w = (z_{k+1}, \dots, z_n)$, $R^k = (R_1, \dots, R_k)$. For every $w \in \Delta_{R^{n-k}}$ the function $f(t, w)$ is holomorphic in $t \in \Delta_{R^k} \setminus \{0\}$, by assumption. Therefore, it extends holomorphically to all of Δ_{R^k} (Hartogs' Theorem 3.5). Fix an arbitrary $0 < \lambda < 1$, set $R^k_\lambda = \lambda R^k$. Then for every $t \in \Delta_{\lambda R^k}$ and $w \in \Delta_{R^{n-k}}$ one has k-dimensional Cauchy formula

$$f(t, w) = (\frac{1}{2\pi i})^k \oint_{|\zeta_1| = \lambda R_1} \dots \oint_{|\zeta_k| = \lambda R_k} \frac{f(\zeta, w)}{\prod_{j=1}^k (\zeta_j - t_j)} d\zeta_k \dots d\zeta_1. \qquad (3.2)$$

The latter subintegral expression is holomorphic in $z = (t, w) \in \Delta_{\lambda R^k} \times \Delta_{R^{n-k}}$. Hence, the integral (3.2) extends the function f holomorphically to the latter product for arbitrary $0 < \lambda < 1$. In particular, the extended function is holomorphic on S. The theorem is proved. $\square$

4 Analytic sets

4.1 Definitions, main properties and Weierstrass Preparatory Theorem

Definition 4.1 An *analytic subset* in a complex manifold M is a subset $A \subset M$ such that each point $p \in A$ has a neighborhood $U = U(p) \subset M$ where there exists a finite collection of holomorphic functions $f_1, \dots, f_k : U \to \mathbb{C}$ such that

$$A \cap U = \{f_1 = \dots = f_k = 0\};$$

in general the number k of functions may depend on the point p.

Remark 4.2 Each analytic subset is closed. Any holomorphic submanifold is an analytic subset, but the converse is not true: the coordinate cross $\{xy = 0\} \subset \mathbb{C}^2$ and the curve $\{y^2 = x^3\} \subset \mathbb{C}^2$ are analytic subsets but not submanifolds.

The theory of analytic sets is one of the basic tools in analysis, complex and algebraic geometry and in many related topics. The analytic subsets behave somewhat line algebraic subsets. They have the following

Main properties of analytic sets.

1. Each analytic subset is a locally finite disjoint union of its regular part (union of those points where it is locally a submanifold) and its complement, the singular part;

2. The regular part is open and dense; the singular part is a smaller analytic subset lying in the closure of the regular part;

3. Each analytic subset is stratified: it is a disjoint union of injectively holomorphically immersed manifolds, which are called *strata*; each stratum is contained either in its regular, or singular part;

4. Each analytic subset is a locally finite union of *irreducible* ones: those that cannot be finite unions of two smaller analytic sets; their regular parts are connected.

These properties will be proved later on for the analytic sets of codimensions one and two. Afterwards we will state a fundamental result of the theory, Remmert Proper Mapping Theorem (without proof).

In the proofs we will use the following obvious corollary of Theorem 3.7.

Corollary 4.3 *Each holomorphic bounded function on a complement of a complex manifold to an analytic subset extends holomorphically to the whole ambient manifold.*

Proof The statement of the corollary is local: the function extends to the analytic subset. Therefore, it suffices to prove it in the case, when the ambient manifold is a polydisk Δ_r and the analytic subset $A \subset \Delta_r$ is the intersection of zero sets $A_j = \{f_j = 0\}$, $f_j : \Delta_r \to \mathbb{C}$ are non-constant holomorphic functions, $j = 1, \dots, k$. Let $f : \Delta_r \setminus A \to \mathbb{C}$ be a bounded holomorphic function. Note that for every j the function f is holomorphic and bounded on the domain $\Delta_r \setminus A_j$, since the latter is contained in $\Delta_r \setminus A$ by the inclusion $A \subset A_j$. Therefore, f extends holomorphically to A_j, and hence, to all of Δ_r, by Theorem 3.7. This proves the corollary. $\square$

First we will prove local versions of the above properties of analytic sets in the case of codimension one, for zero loci of germs of holomorphic function. The proofs are based on the next local theorems saying that each holomorphic function is somewhat like a polynomial in one variable with holomorphic coefficients in the other variables. To state them, let us introduce the following definitions.

Definition 4.4 A polynomial $P_w(z_1) = z_1^d + a_1(w)z_1^{n-1} + \dots + a_0(w)$ with variable coefficients depending holomorphically on $w = (z_2, \dots, z_n)$ from a neighborhood of the origin in $\mathbb{C}^{n-1}$ with $a_j(0) = 0$ is called a *Weierstrass polynomial* in z_1.

Remark 4.5 For every fixed w a Weierstrass polynomial does not vanish identically in z_1 and has the same number d of roots counted with multiplicity.

Definition 4.6 A germ of holomorphic function f at the origin is a *unity,* if $f(0) \neq 0$.

Theorem 4.7 (Weierstrass preparatory theorem). *For every germ f of holomorphic function at the origin in $\mathbb{C}^n$ such that $f(0) = 0$ and $f(z_1, 0) \not\equiv 0$ there exists a unique Weierstrass polynomial $P_w(z_1)$ such that $f(z_1, w) = h(z_1, w)P_w(z_1)$, where h is a unity.*

Proof Fix $\delta > 0$, $r = (r_2, \dots, r_n)$, $r_j > 0$, such that the function f is holomorphic on $\overline{D}_\delta \times \Delta_r$, $f(z_1, 0) \neq 0$ for $z_1 \in \overline{D}_\delta \setminus \{0\}$ and $f|_{\partial D_\delta \times \Delta_r} \neq 0$. Set $g_w(t) = f(t, w)$. The function g_0 has geometrically unique zero in $\overline{D}_\delta$: the origin. Let d denote its multiplicity. Then for every $w \in \Delta_r$ the function g_w has d roots with multiplicities in D_δ and does not vanish on its boundary. Let $b_1(w), \dots, b_d(w)$ denote its roots. The coefficients of the Weierstrass polynomial we are looking for are uniquely determined as the basic symmetric polynomials $\sigma_s = \sigma_s(w)$ in $b_j(w)$ up to sign. (This already proves the uniqueness.) They vanish at $w = 0$ by assumption. Let us show that they are holomorphic functions in w. Indeed, they are expressed as polynomials in the power sums $\hat{\sigma}_s(w) = \sum_j b_j^s(w)$, $s \in \mathbb{N}$. One has

$$\hat{\sigma}_s(w) = \frac{1}{2\pi i} \oint_{\partial D_\delta} \frac{\zeta^s \frac{\partial f}{\partial z_1}(\zeta, w)}{f(\zeta, w)} d\zeta. \tag{4.1}$$

Indeed, the latter integral is equal to the sum of residues of the subintegral expression. The nonzero residues may exist only at those ζ, where $g_w(\zeta) = f(\zeta, w) = 0$. The residue value corresponding to a root ζ of the function $g_w(z_1)$ of multiplicity ν is equal to $\nu\zeta^s$. Indeed, one has

$$g_w(u) = f(u, w) = c(u - \zeta)^\nu (1 + O(u - \zeta)), \text{ as } u \to \zeta; \; c \neq 0,$$

$$\frac{\partial f}{\partial z_1}(u, w) = c\nu(u - \zeta)^{\nu-1}(1 + o(1)) + O((u - \zeta)^\nu) = \frac{\nu}{u - \zeta} f(u, w)(1 + o(1)).$$

This implies that the residue at ζ is equal to $\nu\zeta^s$. This proves (4.1). The right-hand side in (4.1) is holomorphic in $w \in \Delta_r$, since the subintegral expression is holomorphic and its restriction to the integration circle is a uniformly bounded function whenever w run over arbitrary compact subset in Δ_r. Therefore, the integral and hence, the power sums $\hat{\sigma}_s(w)$ are holomorphic on Δ_r. Hence, the elementary symmetric polynomials σ_s are also holomorphic. Therefore, the function

$$P_w(z_1) = \prod_{j=1}^{d}(z_1 - b_j(w)) = z_1^d + \sum_{s=1}^{d}(-1)^s \sigma_s(w) z_1^{d-s}$$

is a Weierstrass polynomial vanishing exactly on the zero set $\Gamma = \{f = 0\}$ of the function f. The ratio $h = \frac{f}{P}$ is a holomorphic function on the complement $(\overline{D}_\delta \times \Delta_r) \setminus \Gamma$. Let us show that it extends holomorphically to Γ and does not vanish there: then the theorem follows immediately. For every fixed w it has a nonzero limit, as z_1 tends to a root of the polynomial $P(z_1, w)$, since the latter root has the same multiplicity for both functions $P_w(z_1)$ and $g_w(z_1)$. Therefore, the function $h(z_1, w)$ is holomorphic in $z_1 \in \overline{D}_\delta$ for every fixed $w \in \Delta_r$. Hence, it can be written as Cauchy integral

$$h(z_1, w) = \frac{1}{2\pi i} \oint_{|\zeta|=\delta} \frac{h(\zeta, w)}{\zeta - z_1} d\zeta, \; z_1 \in D_\delta.$$

The subintegral expression is holomorphic in $(z_1, w) \in D_\delta \times \Delta_r$ and uniformly bounded with derivatives and continuous on compact subsets in

$D_\delta \times \Delta_r$. Therefore, the latter integral, and hence h are holomorphic there. The above limiting argument implies that $h = \frac{f}{P} \neq 0$ on Γ, and hence, is a unity. This proves the theorem. $\square$

4.2 Factorization of holomorphic functions as product of irreducible ones

Definition 4.8 A germ of holomorphic function is said to be *irreducible,* if it is not a product of two holomorphic functions that are not unities. A Weierstrass polynomial is irreducible at zero, if its germ at zero is irreducible as a holomorphic function.

Theorem 4.9 *Each Weierstrass polynomial admits a unique decomposition as a product of irreducible Weierstrass polynomials up to permutation.*

Theorem 4.10 *A germ of holomorphic function f is irreducible, if and only if the subset of points in $\{f = 0\}$ where f has nonzero differential is connected and dense.*

Corollary 4.11 *The ring of germs (local ring) of holomorphic functions at $0 \in \mathbb{C}^n$ is factorial: each function admits a unique decomposition as a product of irreducible ones, up to multiplication by unities and permutations.*

Proof Each function f is a Weierstrass polynomial P in generic local coordinates, where $f(z_1, 0) \not\equiv 0$ up to multiplication by unity (Theorem 4.7). Therefore, it admits a unique presentation as a product of irreducible Weierstrass polynomials $P = P_1 \dots P_k$ (Theorem 4.9). Multiplying some of them by the above unity, we get the desired decomposition of the initial function. Let us now prove the uniqueness. Every factorization $f = f_1 \dots f_k$ yields a factorization $P = P_1 \dots P_k$, where P_j are the Weierstrass polynomials representing f_j. This together with the uniqueness of the latter implies the corollary. $\square$

Theorems 4.9 and 4.10 can be proved by induction in n by using the following well-known facts from algebra:

Gauss Lemma: *if a ring R is factorial, then the corresponding polynomial ring $R[t]$ is also factorial.*

If a ring R is factorial and $u, v \in R[t]$ are coprime, then there exist coprime elements $\alpha, \beta \in R[t]$ and $\gamma \in R$, $\gamma \neq 0$, such that

$$\alpha u + \beta v = \gamma.$$

To make proofs self-contained, we will provide another, geometric argument. To motivate it, let us consider the following two basic examples.

Example 4.12 The Weierstrass polynomial $z_1^2 - z_2^2$ is not irreducible (its germ at the origin is not irreducible): it is the product $(z_1 - z_2)(z_1 + z_2)$ of two irreducible Weierstrass polynomials of degree one. Its zero locus Γ is a union of their zero loci: the lines $\Gamma_\pm = \{z_1 = \pm z_2\}$. The punctured zero locus $\Gamma^0 = \Gamma \setminus \{0\}$ is disconnected and consists of two connected components $\Gamma^0_\pm$: the punctured zero loci of irreducible factors.

Example 4.13 The germ at 0 of the Weierstrass polynomial $z_1^2 - z_2$ is irreducible: it has a nonzero linear part z_2, and hence, cannot be a product of two holomorphic functions vanishing at the origin. Its punctured zero locus Γ^0 is connected: it is the Riemann surface of the square root $z_1 = \sqrt{z_2}$ parametrized bijectively holomorphically by $z_1 \neq 0$. In both this and previous examples the projection of the punctured zero locus Γ^0 to the z_2-space is a double covering.

To make the presentation self-contained, let us recall the definition of covering.

Definition 4.14 A continuous mapping $\pi : S_1 \to S_2$ of topological spaces is a *covering,* if each point $p \in S_2$ has a neighborhood $U = U(p) \subset S_2$ such that $\pi^{-1}(U)$ is a disjoint union of domains $V_r \subset S_1$ that are homeomorphically projected by π onto the domain U. The space S_2 is called the *base* of the covering, and S_1 is called the total (covering) space.

Example 4.15 Let $\pi : S_1 \to S_2$ be a covering, and let S_2 be path connected. Then any two points $p_1, p_2 \in S_2$ have the same number of preimages in S_1 (finite or infinite). This comes from the covering homotopy property: for every path $\alpha : [0,1] \to S_2$ and every point $q \in \pi^{-1}(\alpha(0))$ there exists a unique path $\widetilde{\alpha} : [0,1] \to S_1$ such that $\widetilde{\alpha}(0) = q$ and $\pi \circ \widetilde{\alpha} \equiv \alpha$. If S_2 is a manifold, then S_1 inherits a natural structure of manifold lifted from the base S_2 via the projection.

Remark 4.16 Let $\pi : S_1 \to S_2$ be a covering, and let S_1 and S_2 be oriented manifolds, S_2 being connected. Let the above number of preimages be finite. For every $p \in S_2$ and $q \in \pi^{-1}(p)$ let us count q with the weight "+1" if $d\pi(q) : T_qS_1 \to T_pS_2$ preserves the orientation (its determinant is positive with respect to orienting bases in the latter tangent spaces) and with the weight "-1" otherwise. Then the total sum of weights of preimages of a given point $p \in S_2$ does not depend on p and is called the *degree* of the covering.

Example 4.17 The natural projection $\mathbb{R} \to \mathbb{R}/\mathbb{Z} = S^1$ of the line to the circle is a covering where each point has infinite number of preimages. The mapping $S^1 \to S^1$ induced by any integer homothety $\mathbb{R} \to \mathbb{R}$: $x \mapsto nx$, $n \in \mathbb{Z}$, is a covering of degree n.

Remark 4.18 Each complex manifold carries a natural orientation. Thus, each holomorphic covering over a connected complex manifold has a positive degree. We will be interested in the situation, when the base of a covering is the complement of a connected complex manifold Δ to an analytic subset $A \subset \Delta$. The latter complement is always connected. Indeed, the zero locus of a non-constant holomorphic function is locally the zero locus of a Weierstrass polynomials and has real codimension two. In more detail, the zero locus of a Weierstrass polynomial $P_w(z_1)$ of degree d cuts each z_1-fiber $\mathbb{C} \times \{w\}$ by at most d points. Hence, the z_1-fibers of its complement are locally finitely punctured topological disks and hence, are connected. This together with continuous dependence of the collection of roots on the parameter w easily implies the connectedness of the complement of a polydisk to a zero locus of holomorphic function and hence, the general above connectivity statement.

For the proof of Theorems 4.9 and 4.10 we fix $\delta > 0$, $r = (r_2, \dots, r_n)$, $r_j > 0$, set $\Delta = \Delta_r$, such that $P_w(z_1)$ is holomorphic on $\overline{D}_\delta \times \Delta$ and

$$P_w(z_1) \neq 0 \text{ for every } (z_1, w) \in \partial D_\delta \times \Delta. \tag{4.2}$$

For every $w \in \Delta$ we denote by $k(w)$ the number of geometrically distinct roots of the polynomial $P_w(z_1)$. Set

$$k_{\max} = \max_{w \in \Delta} k(w),$$

$$A = \{w \in \Delta \mid k(w) < k_{\max}\},$$

$$\Gamma = \{P_w(z_1) = 0\} \subset D_\delta \times \Delta, \ \Gamma^0 = \Gamma \setminus \pi^{-1}(A).$$

The proof will be split into the following steps.

Step 1: we prove analyticity of the subset $A \subset \Delta$.

Step 2: we show that the projection $\pi : \Gamma^0 \to \Delta \setminus A$ is a covering of degree $k_{\max}$. We denote $\Gamma_1, \dots, \Gamma_s$ the connected components of the covering space Γ^0. Each point $z = (z_1, w) \in \Gamma_j$ has multiplicity $\mu(z)$: this is the multiplicity of the root z_1 of the polynomial P_w. We show that the latter multiplicity $\mu_j = \mu(z)$ is constant on each Γ_j. Let d_j denote the degree of the covering $\pi : \Gamma_j \to \Delta \setminus A$. For every $w \in \Delta \setminus A$ let $t_{j1}(w), \dots, t_{jd_j}(w)$ denote the z_1-coordinates of the points of intersection $\pi^{-1}(w) \cap \Gamma_j$. Set

$$h_{j,w}(z_1) = \prod_{m=1}^{d_j}(z_1 - t_{jm}(w)).$$

We get that

$$P_w(z_1) = \prod_{j=1}^{s}(h_{j,w}(z_1))^{\mu_j}. \tag{4.3}$$

Step 3: we show that the set Γ^0 is locally defined (in some neighborhood of each its point) as a zero locus of a holomorphic function with nonzero partial derivative in z_1. This together with the covering property (Step 2) implies that each point $w \in \Delta \setminus A$ has a neighborhood $U \subset \Delta \setminus A$ such that $\Gamma^0 \cap \pi^{-1}(U)$ is a disjoint union of graphs of holomorphic functions $z_1(w)$. Hence, the coefficients of the polynomials $h_{j,w}(z_1)$ depend holomorphically on $w \in \Delta \setminus A$. They are bounded, being symmetric functions (up to sign) of some roots of the polynomials $P_w(z_1)$, which are bounded: the roots lie in D_δ for all $w \in \Delta$ by assumption (4.2). Therefore, the latter coefficients are holomorphic on all of Δ (Corollary 4.3). They vanish at 0 by definition, hence, $h_{j,w}(z_1)$ are Weierstrass polynomials.

Step 4: we show that the polynomials $h_{j,w}$ are irreducible. This will easily imply both theorems.

The proof of Step 1 is based on the two following statements on polynomials in one variable.

Theorem 4.19 *Two polynomials $P_1(t)$ and $P_2(t)$ in one variable of degrees d_1 and d_2 respectively have at least l common roots (with multiplicities), if and only if the polynomials*

$$P_1(t), tP_1(t), \dots, t^{d_2-l}P_1(t), P_2(t), \dots, t^{d_1-l}P_1(t)$$

are linearly dependent: that is in the $(d_1+d_2-l+1)\times(d_1+d_2-2l+2)$-matrix of their coefficients all the d_1+d_2-2l+2-minors vanish.

Proof The existence of at least l common roots is equivalent to the fact that the greatest common divisor h of the polynomials P_1 and P_2 has degree at least l. Set $P_j = hq_j$: then q_1 and q_2 are coprime of degrees at most $d_j - l$. The latter statements are equivalent to the equality $P_1q_2 - P_2q_1 = 0$,

$degq_j \leq d_j - l$. The latter is equivalent to linear dependence of the above collection of polynomials. The theorem is proved. $\square$

Remark 4.20 In the case, when $l = 1$, the above matrix of coefficients is square and its determinant equals the resultant of the polynomials P_1, P_2.

Proposition 4.21 *Let $P(t)$ be a polynomial in one variable, d be its degree, and let k be the number of its geometrically distinct roots (i.e., the roots without multiplicity). Then the number of common roots with multiplicities of the polynomial P and its derivative P' equals $d - k$.*

Proof Each root of the polynomial P of multiplicity ν is a root of its derivative with multiplicity $\nu - 1$. Thus, it is their common root of multiplicity $\nu - 1$. Summing up the latter numbers through all the roots yields $d - k$. This proves the proposition. $\square$

Proof of Step 1: analyticity of the set A**.** Note that $k(w) = k_{\max}$ on an open subset in Δ. This follows by definition and from the fact that if we perturb a polynomial continuously, then the number of its geometrically distinct roots may only increase: thus, if we perturb P_{w_0} with $k(w_0) = k_{\max}$ in the family P_w, the number $k(w)$ remains constant. The subset $A \subset \Delta$ coincides with the subset of those points w for which the polynomial P_w and its partial derivative in z_1 have at least $l = d - k_{\max} + 1$ common roots (Proposition 4.21). For every w let $M(w)$ denote the coefficient matrix from Theorem 4.19 corresponding to the latter polynomials and l. Then the set A is defined by vanishing of its $2k_{\max} + 1$-minors (Theorem 4.19). Hence, A is analytic, since the matrix coefficients are holomorphic on Δ. $\square$

Proof of Step 2 and factorization (4.3). Recall that the complement $\Delta \setminus A$ is connected (Remark 4.18). As $w \in \Delta \setminus A$ varies, the roots of the polynomial $P_w(z_1)$ vary continuously and their multiplicities remain unchanged. Indeed, the contrary would imply that some multiple root splits into several ones under perturbation, the number of geometrically distinct roots increases and becomes greater than $k_{\max}$ – a contradiction. This implies that each point $p \in \Delta \setminus A$ has a neighborhood $U \subset \Delta \setminus A$ such that $\pi^{-1}(U) \cap \Gamma^0$ is a union of graphs of continuous functions $z_1(w)$. This proves that the projection $\pi : \Gamma^0 \to \Delta \setminus A$ is a covering of degree $k_{\max}$. Let $\Gamma_1, \dots, \Gamma_s$ denote the connected components of the covering space Γ^0. For every $j = 1, \dots, s$ the points (z_1, w) of the component Γ_j correspond to roots z_1 of the polynomials $P_w(z_1)$ having the same multiplicity μ_j independent on $(z_1, w) \in \Gamma_j$ (constance of multiplicity under deformation, see the above argument). Step 2 is proved, and it implies the factorization (4.3). $\square$

Proof of Step 3: $h_{j,w}(z_1)$ are Weierstrass polynomials with coefficients holomorphic on Δ. Every point $p \in \Gamma_j$ has a small neighborhood where the function $g_j = (P_w(z_1))^{\frac{1}{\mu_j}}$ is holomorphic. Indeed, the latter function is well-defined and holomorphic on the complement of a small neighborhood of the point p to the zero locus $\{P_w(z_1) = 0\}$ (which locally coincides with Γ_j). It is obviously bounded. Therefore, it is holomorphic in the whole neighborhood, including the zero locus (Theorem 3.7). This together with the discussion in Step 3 implies all its statements. $\square$

Now let us prove Step 4: irreducibility of polynomials $h_{j,w}$. It is implied by the following general statement.

Proposition 4.22 *A polynomial $P_w(z_1)$ is irreducible, if and only if $s = 1$ and $\mu_1 = 1$.*

Proof In the case, when either $s \geq 2$, or $\mu_j \geq 2$ for some $j = 1, \dots, s$ the polynomial is obviously not irreducible, since it is equal to a nontrivial product (4.3). Let now $s = 1$, $\mu_1 = 1$. Let us show that P_w is irreducible. Let there exist a holomorphic function $g(z)$ that is a divisor of the function $P_w(z_1)$ different from unity. It suffices to show that $g(z) = P_w(z_1)$ up to multiplication by unity. The zero locus $S = \{g = 0\}$ is non-empty and contained in $\Gamma = \{P_w(z_1) = 0\}$. In particular, g does not vanish identically on the z_1-axis. Therefore, without loss of generality we consider that $g = g_w(z_1)$ is a Weierstrass polynomial. By construction, for every $w \in \Delta \backslash A$ each root of the polynomial $g_w(z_1)$ is a simple root of the polynomial P_w. Hence, it should be simple for g_w as well (since g_w divides P_w). This together with the results of Steps 2, 3 (applied to g_w) implies that the set $S^0 = S \setminus \pi^{-1}(A)$ is a covering over the domain $\Delta \setminus A$ under the projection π. On the other hand, S^0 is contained in a connected covering Γ^0. Hence, $S^0 \subset \Gamma^0$ is an open and closed subset in a connected set Γ^0, and thus, $S^0 = \Gamma^0$. Thus, g_w and P_w have the same roots for $w \in \Delta \setminus A$, and they are simple for both of them. Hence $g_w(z_1) \equiv P_w(z_1)$. The proposition is proved. $\square$

Proof of Theorems 4.9 and 4.10. Theorem 4.10 follows immediately from Proposition 4.22. Let us prove Theorem 4.9.

The above proposition implies that polynomials $h_{j,w}$ in formula (4.3) are irreducible. This yields a factorization of the Weierstrass polynomial P_w under consideration as a product of irreducible ones. The same proposition implies the uniqueness: the complement to $\pi^{-1}(A)$ of the zero locus of an irreducible factor of the polynomial P_w should coincide with some of the sets

Γ_j, and thus, it should coincide with $h_{j,w}$ by the same proposition. Theorem 4.9 is proved. $\Box$

4.3 Implicit Function and Rank Theorems. Weierstrass polynomials in two variables

Theorem 4.23 (Holomorphic Implicit Function Theorem) *Let $F : \mathbb{C}^n \times \mathbb{C}^l \to \mathbb{C}^l$, $(X, Y) \mapsto F(X, Y)$ be a germ of holomorphic mapping at zero, $F(0, 0) = 0$, such that the partial differential $\frac{\partial F}{\partial Y}(0) : T_0\mathbb{C}^l \to T_0\mathbb{C}^l$ is a non-degenerate linear operator. Then there exists a neighborhood $U = V \times W$ of the origin in $\mathbb{C}^n \times \mathbb{C}^l$ such that the intersection $U \cap \{F = 0\}$ is the graph $\{Y = Y(X)\}$ of a holomorphic mapping $Y : W \to V$. Its differential $dY(X)$ at each point X_0, set $Y_0 = Y(X_0)$, is equal to $(\frac{\partial F}{\partial Y})^{-1}(X_0, Y_0)\frac{\partial F}{\partial X}(X_0, Y_0)dX$. That is, the latter matrix product is equal to the Jacobian matrix of the mapping $Y(X)$ at X_0.*

Proof Consider the mapping F as a real mapping $\mathbb{R}^{2n} \times \mathbb{R}^{2l} \to \mathbb{R}^{2l}$, which is a C^∞ mapping, since each holomorphic function is infinitely differentiable. The non-degeneracy of its partial differential in Y is equivalent to the same non-degeneracy statement for the real mapping. The classical Implicit Function Theorem says that for appropriate $U = V \times W$ the intersection $U \cap \{F = 0\}$ is the graph of a smooth mapping $Y : W \to V$, whose differential is equal to the above product. The partial differentials of the function F in the latter product are $\mathbb{C}$-linear, since F is holomorphic. Therefore, the product $dY(X)$ is also $\mathbb{C}$-linear at each point in W, and hence, Y is holomorphic. The theorem is proved. $\Box$

Recall the following definition.

Definition 4.24 A mapping $F : U \to V$ of complex domains (manifolds) is *biholomorphic*, if it is holomorphic and has a holomorphic inverse. A biholomorphic *germ* of mapping $(\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ is defined analogously.

Corollary 4.25 *A germ of mapping $G : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ with non-degenerate differential $dG(0)$ is always biholomorphic.*

Proof It suffices to apply the Implicit Function Theorem to the function $F(X, Y) = G(Y) - X$. $\Box$

Remark 4.26 Each biholomorphic mapping is always a C^∞ diffeomorphism. There exist no biholomorphic mappings of domains of different dimensions, since this is true for diffeomorphisms.

Theorem 4.27 (Constant Rank Theorem). *Let $F : (\mathbb{C}^n, 0) \to (\mathbb{C}^m, 0)$ be a germ of holomorphic mapping at the origin, $F(0) = 0$, and let its differential have constant rank $l \leq m$ in some neighborhood of the origin. Then there exist germs of biholomorphisms (coordinate changes) $g : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$, $h : (\mathbb{C}^m, 0) \to (\mathbb{C}^m, 0)$ such that $h \circ F \circ g(z_1, \dots, z_n) = (z_1, \dots, z_l)$.*

The proof of this theorem repeats the classical proof of the similar theorem from calculus. One has just to notice that the differentials of the mappings g, h constructed in the classical proof are $\mathbb{C}$-linear, and hence, the mappings are holomorphic.

Theorem 4.28 *For every irreducible germ of holomorphic function f at zero, $f(0) = 0$, the regular part of its zero locus $\Gamma = \{f = 0\}$ consists exactly of those points $z \in \Gamma$ where $df(z) \neq 0$.*

Proof We know that each point of the zero locus where the differential is not identically zero is a regular point (Implicit Function Theorem). Let us prove the converse. To do this, we have to show that every point $p \in \Gamma$ where $df(p) = 0$ cannot be a regular point: the germ at p of the zero locus is not a submanifold. Suppose the contrary: it is a submanifold. Then there exist a neighborhood U of the point p in $\mathbb{C}^n$ and local coordinates $z_1, \dots, z_n$ on U such that $U \cap \Gamma = \{z_1 = 0\}$. Therefore, $f(z) = z_1 g(z)$ in U, where $g = \frac{f(z)}{z_1}$ is a holomorphic function on U. One has $g(p) = 0$, since $df(p) = 0$. Note that we already know that $df \neq 0$ on an open and dense subset $\Gamma^0 \subset \Gamma$. Hence, $g(z)$ does not vanish identically on the intersection $U \cap \Gamma$. This implies that the germ Γ_g at p of the zero locus of the function g does not contain the germ of the hyperplane $\{z_1 = 0\}$. On the other hand, the intersection $\Gamma \cap U$ should contains the zero locus of the function g, and hence, contains those its points that do not lie in the latter hyperplane, – a contradiction to the assumption that the latter intersection is contained in the hyperplane. The complement $\Gamma_g \setminus \{z_1 = 0\}$ is indeed non-empty. This can be easily seen if we consider (without loss of generality) that g is a germ at p of Weierstrass polynomial: then for some value of z_2 it obviously has a non-zero root. The theorem is proved. $\square$

Addendum to Theorem 4.28. *Let $f : (\mathbb{C}^n, 0) \to \mathbb{C}$ be a germ of holomorphic function at zero, $f(0) = 0$. Let f be a product of distinct irreducible factors, each of them being taken in power one. Then the singular part of the germ at 0 of the zero locus $A = \{f = 0\}$ coincides with the set of those points p where $df(p) = 0$.*

Proof Let $f = h_1 \dots h_r$, h_j be irreducible germs. For every j let $A_j^0 \subset A$ denote the set of those regular points p of the zero locus $\{h_j = 0\}$ where $h_s(p) \neq 0$ for every $s \neq j$. Set $A^0 = \cup_j A_j^0$. The differential of the function f does not vanish at the points of the set A^0, by definition and Theorem 4.28. Hence $A^0 \subset A_{reg}$, $A \setminus A^0 \supset A_{sing}$. The latter complement $A \setminus A^0$ coincides with A_{sing}: each its point is singular, being either an intersection point of zero loci of distinct irreducible factors, or a singular point of zero locus of an irreducible factor. Therefore the differential of the function f should vanish there. This proves the addendum. □

Now let us study in more detail irreducible germs of functions and Weierstrass polynomials $P_{z_2}(z_1)$ in two variables.

Theorem 4.29 *Let $f(z_1, z_2)$ be an irreducible germ of holomorphic function. Then there exists a germ of holomorphic mapping $g(t)$, $g : (\mathbb{C}, 0) \to (\mathbb{C}^2, 0)$ that maps bijectively a neighborhood of the origin onto a neighborhood of the origin in the zero locus $\{f = 0\}$.*

Proof Without loss of generality we consider that $f = P_{z_2}(z_1)$ is a Weierstrass polynomial, choosing appropriate coordinates. Let d denote its degree. Fix $\delta, r > 0$ such that $P_{z_2}(z_1) \neq 0$ for every $(z_1, z_2) \in \partial D_\delta \times D_r$. Then the d roots of the polynomial $P_{z_2}(z_1)$ with fixed z_2 (which are concentrated at 0, when $z_2 = 0$) do not escape from the disk D_δ, as we deform z_2 from the origin to an arbitrary point in D_r. Set

$$\Gamma = \{P = 0\} \cap D_\delta \times D_r,\ \Gamma^0 = \Gamma \setminus \{0\},\ D_r^0 = D_r \setminus \{0\}.$$

Let $\pi : (z_1, z_2) \mapsto z_2$ be the standard projection. If r is chosen small enough, then the mapping $\pi : \Gamma^0 \to D_r^0$ is a covering of degree d, see Step 2 in the proof of Theorem 4.9. Indeed, here A is an analytic subset of one-dimensional disk $\Delta_r = D_r$. Hence, is a discrete collection of points including the origin, since a zero of a non-constant holomorphic function is always isolated. Taking r small enough, we can achieve that $A = \{0\}$, and hence, $\pi|_{\Gamma^0}$ is a covering. Note that the covering space Γ^0 is a connected manifold, by irreducibility and Theorem 4.10.

In what follows we use the next proposition.

Definition 4.30 Two coverings over the same base are *isomorphic,* if there exists a homeomorphism between their covering spaces that forms a commutative diagram with the projections.

Proposition 4.31 *Every connected covering $\pi : \Gamma^0 \to D_r^0$ of any degree $d > 0$ over punctured disk $D_r^0 \subset \mathbb{C}$ is isomorphic to the d-power $\pi_d : D^0_{r^{\frac{1}{d}}} \to D_r^0$: $t \mapsto t^d$. There are exactly d covering isomorphisms from the latter covering to the former: any two of them are obtained by coordinate change $t \mapsto e^{2\pi i \frac{p}{d}} t$.*

Proof Consider a marked point $s \in D_r^0$ and a closed path $\alpha : [0,1] \to D_r^0$ with base point $\alpha(0) = s$ making one counterclockwise turn around the origin, e.g., going along the circle centered at zero of radius $|s|$. For every $\hat{s} \in \pi^{-1}(s)$ it lifts to a unique path $\hat{\alpha} : [0,1] \to \Gamma^0$, $\alpha(0) = \hat{s}$, $\pi \circ \hat{\alpha} = \alpha$. The mapping $\hat{\alpha}(0) \mapsto \hat{\alpha}(1)$: $\pi^{-1}(s) \to \pi^{-1}(s)$ is well-defined (covering homotopy property) and is a bijection: a permutation of covering preimages. It is a cyclic permutation of order d. Indeed, the fundamental group of the punctured disk is isomorphic to $\mathbb{Z}$ and generated by $[\alpha]$. It should act transitively on the above preimages, by connectivity, and hence, is a cyclic permutation.

Fix a preimage $t_0 \in \pi_d^{-1}(s)$ under the power covering. Consider an arbitrary point t in the punctured disk $D^0_{r^{\frac{1}{d}}}$ and an arbitrary path $\widetilde{\alpha}$: $[0,1] \to D^0_{r^{\frac{1}{d}}}$, $\widetilde{\alpha}(0) = t_0$, $\widetilde{\alpha}(1) = t$. Set $\alpha(u) = (\widetilde{\alpha}(u))^d = \pi_d \circ \widetilde{\alpha}(u)$: this is a path $\alpha : [0,1] \to D_r^0$ starting at s. Let us fix an arbitrary preimage $\tau_0 \in \pi^{-1}(s)$ and consider the lifting to Γ^0 of the path α with the base point τ_0. This yields a path $\hat{\alpha} : [0,1] \to \Gamma^0$, $\hat{\alpha}(0) = \tau_0$, $\pi \circ \hat{\alpha} = \alpha$. Set

$$g(t) = g_{\tau_0}(t) = \hat{\alpha}(1) \in \Gamma^0.$$

The point $g(t)$ depends only on t, and not on $\widetilde{\alpha}$. Indeed, if two paths β_1 and β_2 go from t_0 to t, then the product $\widetilde{\beta} = \beta_1 \beta_2^{-1}$ is a closed path in the covering punctured disk. Therefore, its projection β is homotopic to a power $[\alpha]^m$ that acts by trivial permutation on the preimage $\pi_d^{-1}(s)$. The number m is divisible by d, since the permutation given by $[\alpha]$ is cyclic. This implies that it also acts trivially on the preimages of the other covering π. Equivalently, each lifting to Γ^0 of the path β is a closed path. Equivalently, liftings to Γ^0 of projections of the paths β_1, β_2 with the same base point in Γ^0 have common end. This proves independence of the value $g(t)$ on the choice of path. By construction, $t \mapsto g(t)$ is a covering isomorphism. It depends on the choice of the preimage τ_0. Or equivalently, fix a τ_0 and choose t_0: its choice is unique up to multiplication by a d-th root of unity; we have d possible choices. The latter multiplications are all the automorphisms of the power covering π_d. This implies the statements of the proposition. $\Box$

Let $g : D^0_{r^{\frac{1}{d}}} \to \Gamma^0$ be a covering isomorphism given by the above proposition. It is bijective, extends homeomorphically to the puncture as $g(0) = 0$ and is a holomorphic $\mathbb{C}^2$-valued vector function in the punctured disk, by construction. It extends holomorphically to the puncture 0, since it extends continuously there and by erasing singularity theorem for holomorphic functions in one variable. This proves Theorem 4.29. $\Box$

Corollary 4.32 *Let Γ be the zero locus of an irreducible germ of holomorphic function $f(z_1, z_2)$ at zero, $f(0) = 0$. The tangent lines $T_z\Gamma^0$, $z \in \Gamma^0$ tend to a limit line L through the origin, as $z \to 0$, which we call the tangent line to Γ at the origin. The radial line of a point $z \in \Gamma^0$ also tends to the same limit line L, as $z \to 0$.*

Addendum to the corollary. *In the conditions of the corollary consider the lower homogeneous part of the function f:*

$$f(z) = P_d(z) + O(|z|^{d+1}), \tag{4.4}$$

where $P_d(z)$ is a homogeneous polynomial of degree d in variables $z = (z_1, z_2)$. Then $P_d|_L \equiv 0$.

Proof The parametrization $g(t)$ of zero locus being holomorphic at the origin, it has the form $g(t) = (g_1(t), g_2(t))$, $g_j(t) = c_j t^{p_j}(1 + O(t))$, $c_j \neq 0$, $p_j \in \mathbb{N}$. Choosing appropriate coordinates, we can and will consider that $p_2 > p_1$. The slope $\frac{g_2(t)}{g_1(t)}$ of the radial line through $z = g(t)$ obviously tends to zero, and hence the radial line tends to the z_1-axis. The slope of the tangent line $T_z\Gamma$ is the ratio of derivatives $\frac{g_2'(t)}{g_1'(t))} = \frac{c_2 p_2}{c_1 p_1} t^{p_2 - p_1}(1 + o(1))$. It tends to zero, as $t \to 0$. Therefore, the tangent line tends to the z_1-axis. The corollary is proved.

For the proof of the addendum we have to show that $P_d(z_1, 0) \equiv 0$. Suppose the contrary: the polynomial $P_d(z_1)$ contains the monomial z_1^d with nonzero coefficient, say equal to one. Then, $f(g(t)) = c_1^d t^{dp_1}(1 + o(1))$, as $t \to 0$. Hence, $f(g(t) \neq 0$ for arbitrarily small t and thus, f does not vanish on its zero locus, – a contradiction. The addendum is proved. $\Box$

The next lemma and corollary provide a necessary condition for irreducibility of a function in two variables in terms of its lower Taylor terms.

Lemma 4.33 *Let $f(z_1, z_2)$ be a germ of holomorphic function at zero, $f(0) = 0$, and let P_d be its lower homogeneous part, see (4.4). Let L be a complex line contained in the zero locus $\{P_d = 0\}$. Then there exists at least one* *irreducible component of the zero locus $\{f = 0\}$ that is tangent to L at the origin.*

Proof Let us choose the coordinates (z_1, z_2) so that L is the z_1-axis and P_d does not vanish identically on the z_2-axis. Let μ denote the multiplicity of the line L as a zero line of the polynomial P_d. Set $w = \frac{z_2}{z_1}$, and let us write the functions f and P_d in the coordinates (z_1, w). One has

$$P_d(z_1, z_2) = z_2^\mu \prod_{j=\mu+1}^{d} (z_2 - \alpha_j z_1), \ \alpha_j \in \mathbb{C} \setminus \{0\},$$

$$z_1^p z_2^q = z_1^{p+q} w^q, \ z_1^{-d} P_d(z_1, z_2) = G(w) = w^\mu \prod_{j=\mu+1}^{d} (w - \alpha_j),$$

$$F(z_1, w) = z_1^{-d} f(z_1, z_2) = G(w) + O(z_1). \tag{4.5}$$

In more detail, fix an arbitrary $\delta > 0$. Then the latter asymptotic formula holds, as $|w| \leq \delta$ and $z_1 \to 0$. Fix the above δ to be less than $\min_j |\alpha_j|$. Consider the restriction to $D_\delta \times \{z_1\}$ of the functions F and G.

Claim. *The function $F(z_1, w)$ in w with fixed z_1 has μ zeros in D_δ, whenever z_1 is small enough.*

Proof The only zero of the function $G(w)$ in $\overline{D}_\delta$ is the center of the disk, which has multiplicity μ. The function $G(w)$ does not vanish on ∂D_δ and thus, the increment of its argument along the boundary equals $2\pi d$. This together with (4.5) implies that the latter statement holds for the function $F(z_1, w)$ in w with fixed z_1, whenever z_1 is small enough. This together with the Argument Principle applied to F proves the claim. □

The claim holds for δ being arbitrarily small. This implies that there exists a sequence of points $p_n \in \{f = 0\}$ converging to the origin so that their radial lines tend to the z_1-axis: the line L. This together with Corollary 4.32 implies that there exists an irreducible component of the zero locus of the function f that is tangent to L at the origin. This proves the lemma. □

Corollary 4.34 *Let a germ at 0 of function $f(z_1, z_2)$, $f(0) = 0$ be irreducible. Then the zero locus of its lower homogeneous part P_d, see (4.4), consists of just a unique complex line through the origin.*

Proof If the lower homogeneous part vanishes on at least two distinct lines, then we would have at least two distinct irreducible components of the zero locus $\{f = 0\}$ tangent to them. This implies that f cannot be irreducible. The corollary is proved. □

4.4 Irreducibility of analytic sets

In the previous subsections we have been mostly discussing irreducibility of germs of holomorphic functions. Here we discuss irreducibility of (germs of) analytic sets and show that zero locus of a germ of irreducible function is irreducible.

Proposition 4.35 *Every connected complex manifold is an irreducible analytic set.*

Proof Suppose the contrary: let M be a connected complex manifold that is a union of two distinct analytic subsets $A_1, A_2 \neq M$. They are closed subsets in M. We claim that $IntA_j = \emptyset$. Indeed, let, by contradiction, $IntA_1 \neq \emptyset$, $p \in \partial A_1$. Let U be a connected neighborhood of the point p where $A_1 \cap U$ is defined as zero locus of collections of holomorphic functions on U. Thus, there exists a non-constant holomorphic function $f : U \to \mathbb{C}$ that vanishes on $A_1 \cap U$: $A_1 \cap U \neq U$, since $p \in \partial A_1$. The nonconstant function f vanishes on a nonempty open subset $IntA_1 \cap U$, – a contradiction to the uniqueness of analytic extension. Hence, A_j are closed nowhere dense subsets in M, and their complements $M \setminus A_j$ are open and dense. Therefore, the intersection $M \setminus (A_1 \cup A_2)$ is open and dense, and hence, is non-empty. The contradiction thus obtained proves the proposition. □

General fact: For every analytic set $A \subset M$ the set A_{reg} consists of its regular points: those points where A is locally a submanifold. That is for every $p \in A_{reg}$ there exists a neighborhood $U = U(p) \subset M$ such that there exist local coordinates $(z_1, \dots, z_n)$ on U in which $A \cap U = \{z_1 = \dots = z_j = 0\}$; the number j is called the *local codimension* of the set A at the point p; the number $n - j$ is called its *local dimension. The subset $A_{reg} \subset A$ is open and dense.*

The openness of the set A_{reg} is obvious. Its density is a theorem, which we will use but will not prove in the cours.

Theorem 4.36 *An analysis set A is irreducible, if and only if A_{reg} is connected.*

Proof that connectivity implies irreducibility. Suppose the contrary: A is a union of two analytic sets A_1, and A_2; $A_j \neq A$. None of them contains A_{reg}: if $A_j \supset A_{reg}$, then $A_j = A$, since A_j is closed and A_{reg} is dense. The subsets $B_j = A_j \cap A_{reg} \subset A_{reg}$ are analytic subsets of the connected manifold A_{reg}, $B_j \neq A_{reg}$, and their union is A_{reg}, – a contradiction to the above proposition. This proves the implication. □

We have been already working with germs of analytic subsets (germs of hypersurface: zero locus of germ of function). Let us recall the general notion of germ of a subset.

Definition 4.37 Let M be a topological space, $p \in M$. Two subsets $A, B \subset M$ are *p-equivalent,* if there exists a neighborhood $U = U(p) \subset M$ where $A \cap U = B \cap U$. A *germ of subset* at p is a class of p-equivalent subsets. The germ at p of a subset A will be denoted (A, p). A union of two germs $(A, p) \cup (B, p)$ is the germ at p of the union $A \cup B$. A *germ of analytic subset* at a point p in a complex manifold M is a germ of analytic subset A defined in some neighborhood of the point p. A germ of subset (A, p) is *connected*, if it has a connected representative in arbitrarily small neighborhood of the point p.

Definition 4.38 A germ of analytic subset (A, p) is *irreducible,* if it is not a union of two germs $(A_1, p) \cup (A_2, p)$, $(A_j, p) \neq (A, p)$. Or equivalently, if it has an irreducible representative: an irreducible analytic subset in arbitrarily small neighborhood of the point p.

Corollary 4.39 *A germ of analytic set* (A, p) *is irreducible, if and only if* (A_{reg}, p) *is connected.*

The corollary immediately follows from the above theorem.

Now let us recall what was already proved for a germ $(\Gamma, 0)$ of zero locus

$$\Gamma = \{f = 0\}$$

of a germ of non-constant holomorphic function f, $f(0) = 0$.

Consider the coordinates (z_1, w), $w = (z_2, \dots, z_n)$, where the function f is represented by a Weierstrass polynomial $P_w(z_1)$. Fix a product $D_\delta \times \Delta_r$, $D_\delta \subset \mathbb{C}$, $\Delta_r \subset \mathbb{C}^{n-1}$, such that f is holomorphic on $\overline{D}_\delta \times \Delta_r$,

$$f_{D_\delta \times \{0\}} \text{ has unique zero at the origin and } f|_{\partial D_\delta \times \Delta_r} \neq 0.$$

Let $\pi : (z_1, w) \mapsto w$ denote the standard projection. There exists an analytic subset $A \subset \Delta_r$ such that

$$\pi : \Gamma^0 = \Gamma \setminus \pi^{-1}(A) \to \Delta_r \setminus \pi^{-1}(A) \text{ is a covering of some finide degree } d.$$

The covering space has a natural structure of complex manifold, for which the inclusion $\Gamma^0 \to \mathbb{C}^n$ is holomorphic. It is open and dense subset in Γ that is contained in Γ_{reg}.

1) *A germ f is irreducible, if and only if the covering manifold Γ^0 is connected. In this case $\Gamma^0 = \Gamma_{reg}$.*

Corollary 4.40 *The zero locus of a germ of irreducible holomorphic function is a germ is of irreducible analytic set.*

2) *Let f be an irreducible germ. Then the singular part $\Gamma_{sing} = \Gamma \setminus \Gamma_{reg}$ of its zero locus is an analytic set:* it is defined by equations

$$\begin{cases} f = 0 \\ \frac{\partial f}{\partial z_j} = 0 \text{ for every } j = 1, \dots, n \end{cases},$$

by Theorem 4.28.

3) *Each germ f is a product*

$$f = \prod_{j=1}^{s} h_j^{\mu_j}, \ h_j \text{ are irreducible.}$$

This factorization is unique up to multiplication by unities and permutations. This yields a decomposition

$$\Gamma = \cup_{j=1}^{s} \Gamma_s, \ \Gamma_j = \{h_j = 0\},$$

the germs Γ_j are irreducible.

Proposition 4.41 *The above set $A \subset \Delta_r$ is a hypersurface: the zero locus of a holomorphic function on Δ_r.*

Proof Recall that the set A is defined by the condition on w saying that the polynomial $P_w(z_1)$ has more root collisions than typically.

Case 1): the polynomial $P_w(z_1)$ is irreducible. Then for a typical w the polynomial $P_w(z_1)$ has simple roots. Thus, the set A corresponds to those w for which it has at least one multiple root, that is, a common root with its partial derivative in z_1. Hence, A is the zero locus of the resultant of the polynomial and its derivative in z_1. The resultant is holomorphic in $w \in \Delta_r$, being polynomial in the coefficients of $P_w(z_1)$.

Case 2): the polynomial $P_w(z_1)$ is the product of powers of irreducible polynomials $h_{j,w}(z_1)$, $j = 1, \dots, s$. For every j let $A_j \subset \Delta_r$ denote the above analytic subset corresponding to the polynomial $h_{j,w}$: it is the zero locus of the resultant of the latter polynomial and its derivative in z_1. For every $k, l = 1, \dots, s$, $k \neq l$ let A_{kl} denote the zero locus of the resultant of the polynomials $h_{k,w}(z_1)$ and $h_{l,w}(z_1)$. Then one has $A = \cup_j A_j \cup (\cup_{k \neq l} A_{kl})$, and it is the zero locus of the product of all the above resultants. The proposition is proved. $\square$

Theorem 4.42 *Let A be a (germ of) analytic set. Then it admits a unique presentation as a union $A = \cup_{j=1}^{s} A_j$ of irreducible ones.*

We will not present a proof of the existence: we will prove only the uniqueness.

Proof of the uniqueness. Let $A = B_1 \cup B_2$, $B_j \neq A$. It suffices to show that each B_j is a union of some A_k. Suppose the contrary: say, the intersection $B_1 \cap A_k$ is non-empty and smaller than A_k. Then $A_k = C_1 \cup C_2$, $C_j = B_j \cap A_k \neq A_k$ are analytic sets, – a contradiction to the irreducibility of the subset A_k. The uniqueness is proved. □

Definition 4.43 The *dimension* of a (germ of) irreducible analytic set A is the dimension of its regular part (which is a connected manifold). The dimension of an arbitrary analytic set A at its point p is the maximal dimension of irreducible component of its germ at p. Or equivalently, it is the maximal of the dimensions of irreducible components containing p. If all the irreducible components of an analytic set have dimension k, then we say that A is of *pure dimension* k.

4.5 Covering presentation and Proper Mapping Theorem

Here we discuss covering presentation of analytic subsets in general. Then we state one of the key results of the theory of analytic sets: Remmert Proper Mapping Theorem.

Theorem 4.44 *Let S be a germ of analytic set of pure dimension k at the origin in $\mathbb{C}^n$. Then its projection to a generic k-dimensional subspace E can be presented as a covering of finite degree over a polydisk $\Delta \subset E$ that is ramified over an analytic hypersurface: zero locus of a holomorphic function on Δ.*

We have already proved the theorem in the case of codimension one: for zero loci of holomorphic functions. We have shown that the projection is a covering of finite degree that is ramified (has colliding preimages) over an analytic subset $A \subset \Delta$ that is a zero locus of holomorphic function, see Proposition 4.41.

Now let us prove the theorem in the case of codimension two, more precisely, when S is the intersection of zero loci of two coprime irreducible holomorphic functions.

Proof in codimension two (sketch). Let $P_{1,w}(z_1)$, $P_{2,w}(z_1)$ be distinct irreducible Weierstrass polynomials. Set

$$\Gamma_j = \{P_{j,w}(z_1) = 0\}, \ S = \Gamma_1 \cap \Gamma_2.$$

Then the polynomials are coprime. Indeed, otherwise $P_{1,w}(z_1)$ should vanish on an open subset of the connected manifold $\Gamma_{2,reg}$, and hence, on all of Γ_2, and the same should hold for interchanged polynomials. Then $P_{1,w}$ and $P_{2,w}$ divide each other, and hence, are equal, being monic polynomials in z_1, – a contradiction.

Consider the *resultant* $\gamma(w)$ of polynomials $P_{1,w}(z_1)$, $P_{2,w}(z_1)$ as polynomials in one variable z_1. Let d_j be their degrees in z_1. The resultant is obviously a polynomial function of their coefficients, and hence, is holomorphic in w. The projection $\pi : (z_1, w) \mapsto w$ maps S onto the zero locus $\Sigma = \{\gamma = 0\}$, and each point in Σ has a finite number of preimages, no greater than $\min_j d_j$. Now we can represent γ as a Weierstrass polynomial, say, in z_2 choosing appropriate coordinates. Then, its zero locus will be projected with finite degree onto a polydisk in some $n-2$-dimensional subspace. Finally, the composition of the above projections maps S with a finite degree onto the latter polydisk. The ramification points (where some preimages collide) correspond to those of the projection of the zero locus of the resultant (which form a hypersurface by Proposition 4.41), and to those points of the set Σ over which the polynomials $P_{j,w}(z_1)$ have more common roots than "typically". This implies the statement of the theorem for the analytic subset S, modulo the fact that the latter ramification points are projected to a hypersurface. $\square$

Definition 4.45 A mapping $f : V \to W$ of topological spaces is *proper,* if the preimage of every compact subset in W is a compact subset in V.

Theorem 4.46 (Remmert Proper Mapping Theorem) *Let $f : M \to N$ be a holomorphic mapping of complex manifolds. Let $A \subset M$ be an analytic subset. Let the restriction $f|_A$ be a proper mapping. Then the image $f(A) \subset N$ is an analytic subset.*

The proof of this theorem requires very powerful tools of complex analysis, and we will not discuss it here.

Remark 4.47 The general properties of analytic sets discussed before can be proved by using Proper Mapping Theorem.

Corollary 4.48 *Let M, R be complex manifolds, and let R be compact. Then the projection to M of every analytic subset in the product $M \times R$ is an analytic subset in M.*

Corollary 4.49 *Let $P_{1,w}(z_1), \dots, P_{k,w}(z_1)$ be an arbitrary collection of Weierstrass polynomials that are holomorphic functions on $\mathbb{C} \times \Delta_r$. Then the intersection of their zero loci is projected onto an analytic subset in Δ_r.*

5 Introduction to complex algebraic geometry. Chow Theorem

Definition 5.1 A subset $A \subset \mathbb{C}^n$ is called *affine algebraic,* if

$$A = \{P_1 = \dots = P_k = 0\} \ P_j \text{ are polynomials.}$$

Remark 5.2 If A is an infinite intersection of zero loci of polynomials, then one can always choose a finite number of polynomials defining A. That is, the ideal consisting of polynomials vanishing on A has a finite basis (Hilbert's theorem).

We will concentrate on algebraic subsets in projective spaces. Let us recall what is the projective space. The group $\mathbb{C}^*$ acts on $\mathbb{C}^{n+1} \setminus \{0\}$ by multiplication of vectors by complex numbers. The quotient of this action is called the *projective space* $\mathbb{CP}^n$. It is just the set of all the complex lines through the origin in $\mathbb{C}^{n+1}$. Each point $z = (z_0, \dots, z_n) \in \mathbb{C}^{n+1} \setminus \{0\}$ represents a point of the projective space denoted $(z_0 : z_1 : \dots : z_n)$. The latter presentation is called "homogeneous coordinates", which are uniquely defined by the point up to common constant factor.

The projective space $\mathbb{CP}^n$ is a complex manifold equipped with affine atlas. For every $j = 0, \dots, n$ the corresponding affine chart is the subset defined by the inequality $z_j \neq 0$. It consists of points with homogeneous coordinates $(z_0 : z_1 : \dots z_{j-1} : 1 : z_{j+1} : \dots : z_n)$ and is naturally biholomorphic to $\mathbb{C}^n$ equipped with the coordinates $(z_0, \dots, z_{j-1}, z_{j+1}, \dots, z_n)$. The above affine charts cover all of $\mathbb{CP}^n$, and the transitions between them are invertible rational mappings on their intersections. For example, the transition of the chart number 0 with coordinates $(z_1, \dots, z_n)$ to the chart number 1 with coordinates $\zeta_0, \zeta_2, \dots, \zeta_n)$ is defined by the formula

$$\zeta_0 = \frac{1}{z_1}, \zeta_2 = \frac{z_2}{z_1}, \dots, \zeta_n = \frac{z_n}{z_1}.$$

The standard action of the group $GL_{n+1}(\mathbb{C})$ on $\mathbb{C}^{n+1}$ induces the action of the group $PGL_{n+1} = GL_{n+1}/\mathbb{C}^*$ on $\mathbb{CP}^n$ by projective transformations. Let $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ denote the standard projection.

Let $P(z_0, \dots, z_n)$ be a homogeneous polynomial. Its zero locus

$$\widetilde{\Gamma}_P = \{P = 0\} \subset \mathbb{C}^{n+1} \setminus \{0\}$$

is $\mathbb{C}^*$-invariant. Its projection image will be denoted

$$\Gamma_P = \pi(\widetilde{\Gamma}_P) \subset \mathbb{CP}^n.$$

Definition 5.3 A *projective algebraic subset* in $\mathbb{CP}^n$ is an intersection

$$A = \cap_{j=1}^{l} \Gamma_{P_s} = \{P_1 = \dots = P_l = 0\} \subset \mathbb{CP}^n.$$

Remark 5.4 The intersection of a projective algebraic subset with an affine chart, say number j, is an affine algebraic subset there. It is defined as zero locus of the system of polynomials $P_s(z_0, z_1, \dots, z_{j-1}, 1, z_{j+1}, \dots, z_n)$. Vice versa, the closure in $\mathbb{CP}^n$ of each affine algebraic subset in $\mathbb{C}^n$ is a projective algebraic subset.

Remark 5.5 Algebraic sets (both affine and projective) are obviously analytic. For the affine sets the converse is false. For example, the graph $\{y = e^x\} \subset \mathbb{C}^2$ is not algebraic, since no polynomial vanishes on it. The next theorem says that for the projective sets the converse is true.

Theorem 5.6 (Chow) *Each analytic subset in* $\mathbb{CP}^n$ *is algebraic.*

The proof of Chow Theorem will be done first for hypersurfaces, see the next definition. This part of the proof is self-contained. Afterwards the proof in the general case will be done by induction in codimension using Proper Mapping Theorem.

Definition 5.7 A subset $A \subset M$ in a complex manifold M is a *hypersurface,* if each point $p \in A$ has a neighborhood $U = U(p) \subset M$ such that there exists a holomorphic function $f : U \to \mathbb{C}$ for which $A \cap U = \{f = 0\}$.

Remark 5.8 An analytic subset A of pure codimension one is always a hypersurface. Indeed, this is a local statement: given a point $p \in A$, we have to show that the germ (A, p) is given by a zero locus of germ of holomorphic function. Fix a non-constant germ of holomorphic function that vanishes on A: it exists by analyticity. Let $h_1, \dots, h_k$ denote those of its irreducible factors that vanish on some open subsets in A. We claim that the product $H = \prod_j h_j$ vanishes exactly on A. It is easy to see that $H \equiv 0$ on A. Now it suffices to show that each germ of holomorphic function vanishing on A is divisible by h_j, or equivalently, vanishes on its zero locus. Or equivalently, vanishes on some open subset of its zero locus (which has connected regular part). This follows from the fact that some open subset of zero locus of each h_j is contained in A, by construction and since A is of pure codimension one.

In the proof of Chow Theorem and in what follows we will use the following property of hypersurfaces and notion of intersection index of a hypersurface and a one-dimensional submanifold.

Definition 5.9 Let (A, p) be a germ of analytic subset in a complex manifold. The *ideal* I_A of the set A is the ideal in the local ring of germs of holomorphic functions at p consisting of those ones that vanish on A. Recall that an ideal in a ring is said to be the *principal* one, if it is generated by a single element H: each element of the ideal is divisible by H.

Proposition 5.10 *The ideal corresponding to a germ of hypersurface is principal.*

Proof Let (A, p) be a germ of hypersurface: it is zero locus of germ of some holomorphic function f. Then

$$f = \prod_{j=1}^{l} h_j^{d_j},$$

$h_1, \dots, h_l$ are distinct irreducible polynomials. Every function g vanishing on A vanishes on the zero loci of the polynomials h_j, which are contained in A. Hence, g is divisible by the product

$$H = \prod_{j=1}^{l} h_j.$$

Finally, each germ of holomorphic function vanishing on A is divisible by H. The proposition is proved. □

Definition 5.11 Let (A, p) be a germ of hypersurface in a complex manifold. Let (Γ, p) be a germ of one-dimensional complex submanifold (that is, a regular analytic curve) that is not contained in (A, p). Let H be the germ of function generating the ideal I_A, see the above proposition. The restriction to Γ of the function H is a function in one variable having isolated zero at p. Its multiplicity as that of the root of the function $H|_\Gamma$ is called the (local) *intersection index* of the germs (A, p) and (Γ, p).

Example 5.12 The intersection index of transverse germs of complex submanifolds (regular hypersurface and curve) is always equal to one.

Proposition 5.13 *Let A and Γ be as above. The every other germ of one-dimensional submanifold Γ' C^1-close to Γ intersects A at a finite number of points close to p, and the sum of their intersection indices is equal to the intersection index of the germs (A, p) and (Γ, p). In particular, if $\Gamma' \cap A \subset A_{reg}$ and the latter intersection is transverse at each its point, then the number of their intersection points is equal to the above intersection index.*

Proof Given a holomorphic function in one variable with isolated zero at a given point p of a given multiplicity d, every close holomorphic function (uniformly close on some compact neighborhood of the point p) has d zeros close to p. This implies the statement of the proposition. □

Example 5.14 Consider a Weierstrass polynomial $P_w(z_1)$ of degree d that is a product of distinct irreducible ones. Then the intersection index of its zero locus with the line $\{w = 0\}$ equals d.

Definition 5.15 The (global) *intersection index* of a hypersurface with a connected one-dimensional complex submanifold is the sum of the local intersection indices at all their intersection points.

Remark 5.16 The number of the above intersection points is finite. This follows from the fact that a non-constant function of one variable holomorphic on a neighborhood of a point p cannot have a sequence of zeros accumulating to p. The global intersection index is invariant under continuous deformations of the one-dimensional complex submanifold, by Proposition 5.13. In particular, if the ambient manifold is $\mathbb{CP}^n$, then the intersection index remains the same if we replace the given hypersurface and submanifold by their images under arbitrary projective transformations for which the image of the latter is not contained in that of the former.

5.1 Proof of Chow Theorem for hypersurfaces

First let us recall what is the projection from a point in $\mathbb{CP}^n$. Fix a point $p \in \mathbb{CP}^n$ and a hyperplane $\mathbb{CP}^{n-1} \subset \mathbb{CP}^n \setminus \{p\}$. The projection

$$\pi_p : \mathbb{CP}^n \setminus \{p\} \to \mathbb{CP}^{n-1}$$

sending a point q to the intersection with $\mathbb{CP}^{n-1}$ of the line pq is called the *projection from the point p.*

Example 5.17 Let $p = (0 : 1 : 0 : \dots : 0)$ in the homogeneous coordinates, and let $\mathbb{CP}^{n-1}$ be the hyperplane $\{z_1 = 0\}$. Consider the affine chart "number 0": $\mathbb{C}^n = \{z_0 = 1\}$ with the coordinates $(z_1, \dots, z_n)$. The point p is thus the intersection point of the z_1-axis with the infinite projective hyperplane $\{z_0 = 0\}$. The projection from the point p in the above affine coordinates takes the form $\pi_p : (z_1, w) \mapsto w$, $w = (z_2, \dots, z_n)$.

Let $\Gamma \subset \mathbb{CP}^n$ be a hypersurface. Fix a point $p \in \mathbb{CP}^n \setminus \Gamma$. Without loss of generality we consider that $p = (0 : 1 : \dots : 0)$ (applying appropriate projective transformation). The proof of Chow Theorem consists of the two following steps.

Step 1. We show that the intersection of the hypersurface Γ with the above affine chart is the zero locus

$$\Gamma \cap \mathbb{C}^n = \{P_w(z_1) = 0\}, P_w(z_1) = z_1^d + \sum_{j=1}^{d} a_j(w) z_1^{d-j},$$

where $a_j(w)$ are holomorphic functions on $\mathbb{C}^n$ that admit a polynomial upper bound of degree j: there exists a constant $C > 0$ such that

$$|a_j(w)| < C(|w|^j + 1). \tag{5.1}$$

Step 2. We show that bounds (5.1) imply that $a_j(w)$ are polynomials of degrees at most j, and hence, $P_w(z_1)$ is a polynomial of degree d. Thus, $\Gamma \cap \mathbb{C}^n$ is an affine algebraic set, and the homogenization of the polynomial $P_w(z_1)$ is a homogeneous polynomial vanishing on Γ. In the affine chart, it vanishes exactly on Γ. It does not vanish identically on the infinite hyperplane $\{z_0 = 0\}$. This easily implies that its zero locus coincides with Γ: each irreducible component of a germ of its zero locus intersects the affine chart, the intersection lies in Γ, and hence, Γ contains the component.

Proof of Step 1. Let d denote the intersection index of the hypersurface Γ and a complex projective line (say, the z_1-axis) that is not contained in it.

For every $w \in \mathbb{C}^{n-1}$ let $t_1(w), \dots, t_{l(w)}(w)$ denote the z_1-coordinates of the points of intersection of the hypersurface Γ with the line $L_w = \mathbb{C} \times \{w\}$. Let $d_j(w)$ denote their local intersection index at $(t_j(w), w)$: one has

$$d = \sum_j d_j(w).$$

Set

$$P_w(z_1) = \prod_{j=1}^{l(w)} (z_1 - t_j(w))^{d_j(w)}.$$

It is a polynomial of degree d in z_1. Locally this is the product of Weierstrass polynomials defining Γ in neighborhoods of the points $(t_j(w), w)$: generating the corresponding ideals. Hence, it is holomorphic in $(z_1, w) \in \mathbb{C}^n$.

Proposition 5.18 *For every $w \in \mathbb{C}^{n-1}$ set*

$$\alpha(w) = \max\{|z_1| \ |(z_1, w) \in \Gamma)\} > 0.$$

There exists a constant $c > 0$ such that

$$\alpha(w) < c(||w|| + 1) \text{ for all } w \in \mathbb{C}^{n-1}. \tag{5.2}$$

Proof Suppose the contrary: there exists a sequence of points $p_k = (t_k, w_k) \in \Gamma$ such that

$$\frac{|t_k|}{||w_k|| + 1} \to \infty, \text{ as } k \to \infty; \text{ thus, } t_k \to \infty. \tag{5.3}$$

Then $p_k \to p$. Indeed, in the homogeneous coordinates

$$p_k = (1 : t_k : w_k) = (1 : t_k : z_{2k} : \dots : z_{nk}) = (\frac{1}{t_k} : 1 : \frac{z_{2k}}{t_k} : \dots : \frac{z_{nk}}{t_k}).$$

All the latter homogeneous coordinates except for the second (unity) tend to zero, by (5.3). Hence, $p_k \to p$. This together with the closedness of the set Γ and the inclusion $p_k \in \Gamma$ implies that $p \in \Gamma$. The contradiction thus obtained proves the proposition. □

One has

$$P_w(z_1) = z_1^d + \sum_{j=1}^{d} a_j(w) z_1^{d-j}.$$

Each $a_j(w)$ is a holomorphic function on $\mathbb{C}^{n-1}$, since $P_w(z_1)$ is holomorphic on $\mathbb{C}^n$. There exists a $C > 0$ such that inequalities (5.1) hold, by (5.2) and since $a_j(w)$ is a polynomial of degree j in the roots of the polynomial $P_w(z_1)$. Step 1 is proved. □

Proof of Step 2. The polynomiality of the functions $a_j(w)$ follows immediately from their upper bounds (5.1) and the following general proposition.

Proposition 5.19 *Let $a : \mathbb{C}^n \to \mathbb{C}$ be a holomorphic function with at most polynomial growth at infinity: that is, there exist a $d \in \mathbb{N}$ and a constant $C > 0$ such that*

$$|a(z)| < C(||z||^d + 1) \text{ for all } z \in \mathbb{C}^n. \tag{5.4}$$

Then $a(z)$ is a polynomial of degree at most d.

Proof Induction in n.

Induction base: $n = 1$. Then a is an entire function of one variable having pole of order at most d at infinity. Its Taylor series is its Laurent series at infinity. Hence, it should not contain powers bigger than d. Thus, a is a polynomial of degree at most d.

Induction step. Let the statement of the proposition be proved for $n = k$. Let us prove it for $n = k + 1$. Denote $z = (z_1, w)$, $w = (z_2, \ldots, z_n)$. For every fixed $w \in \mathbb{C}^{n-1}$ the function $a(z_1, w)$ in one variable z_1 is a polynomial of degree at most d (induction base). Hence,

$$a(z) = \sum_{j=0}^{d} a_j(w) z_1^j.$$

The functions $a_j(w)$ are holomorphic on $\mathbb{C}^{n-1}$. We claim that they satisfy polynomial bound (5.4) with appropriate $C > 0$. Indeed, fix arbitrary distinct $t_0, \ldots, t_d \in \mathbb{C}$. There exists a constant $C > 0$ such that for every $j = 0, \ldots, s$ the function $a(t_j, w)$ in w satisfies polynomial bound (5.4), by assumption. The vector function $(a(t_0, w), \ldots, a(t_d, w))$ is obtained from $(a_0(w), \ldots, a_d(w))$ by multiplication by the van der Monde matrix in $(t_0, \ldots, t_d)$, which is non-degenerate and independent on w. The components of the former vector function satisfy polynomial bound (5.4), hence so do the components of the latter. This together with the induction hypothesis applied to the functions $a_j(w)$ implies that they are polynomial of degree at most d. Finally, the function $a(w)$ is a polynomial of degree at most $2d$. It cannot have degree greater than d, by (5.4). This proves the proposition.
□

Step 2 is proved. The proof of Chow Theorem for hypersurfaces is complete. $\square$

5.2 Proof of Chow Theorem for arbitrary analytic subset

It suffices to prove Chow Theorem for an irreducible analytic subset $\Gamma \subset \mathbb{CP}^n$. The proof will be done by induction in the codimension.

Induction base: $codim\Gamma = 1$. Then Γ is a hypersurface, see Remark 5.8, and it is algebraic by Chow Theorem for hypersurfaces proved above.

Induction step. Let we have proved Chow Theorem for irreducible analytic subsets of codimension at most k. Let us prove it in the case of codimension $k+1$. Thus, let $codim\Gamma = k+1 \geq 2$.

Fix an arbitrary point $q \in \mathbb{CP}^n \setminus \Gamma$. It suffices to show that there exists a homogeneous polynomial $P_q(z_0 : \dots : z_n)$ such that $P_q|_\Gamma \equiv 0$ and $P_q(q) \neq 0$, that is, P_q does not vanish on the line represented by q, i.e., does not vanish on the collection of homogeneous coordinates of the point q. Therefore, Γ is the intersection of the zero loci of polynomials P_q constructed for all $q \notin \Gamma$. The ideal of homogeneous polynomials vanishing on Γ has a finite basis, as every ideal, by Hilbert Basis Theorem. The common zero locus of its basis coincides with Γ, since the ideal contains all P_q. Hence, Γ is algebraic.

Fix a point $\beta \in \Gamma_{reg}$. Fix a point $p \in \mathbb{CP}^n \setminus (\Gamma \cup \{q\})$ satisfying the following conditions:

1) the projective line $L = pq$ is disjoint from the set Γ;

2) the line $p\beta$ is not contained in the tangent subspace $T_\beta\Gamma_{reg}$.

Both conditions hold for an open and dense subset of points $p \in \mathbb{CP}^n$, and hence, can be both achieved. For the second condition this is obvious. For the first condition this holds, since $codim\Gamma \geq 2$. In more detail, if L intersects Γ, it intersects it in a finite number of points, since $q \notin \Gamma$, as in Remark 5.16. One can achieve by deformation that the intersection points be regular. Then each regular intersection point can be erased by arbitrarily small deformation of the line L, since $codim\Gamma \geq 2$.

Consider the projection $\pi_p : \mathbb{CP}^n \setminus \{p\} \to \mathbb{CP}^{n-1}$ from the point p. The image $\pi_p(\Gamma) \subset \mathbb{CP}^{n-1}$ is an analytic subset, by Proper Mapping Theorem. Its dimension equals $dim\Gamma$, since the germ of projection $\pi_p : \Gamma \to \mathbb{CP}^{n-1}$ at β is an immersion, i.e., has non-degenerate differential, by Condition 2). Therefore, the image $\pi_p(\Gamma) \subset \mathbb{CP}^{n-1}$ has codimension $codim\Gamma - 1$. Hence, $\pi_p(\Gamma)$ is algebraic, by induction hypothesis. Let us choose the homogeneous coordinates $(z_0 : \dots : z_n)$ on $\mathbb{CP}^n$ so that $p = (0 : 1 : 0 : \dots : 0)$ and $\mathbb{CP}^{n-1}$ is the coordinate projective $(z_0 : z_2 : \dots : z_n)$-hyperspace. Then in the affine chart $z_0 = 1$ with the coordinates $(z_1, \dots, z_n)$ $\pi_p : (z_1, w) \mapsto w$

is the projection along the z_1-axis. Thus, $\alpha = \pi_p(q) \notin \pi_p(\Gamma)$ (Condition 1)), and there exists a homogeneous polynomial $P(z_0 : z_2 : \cdots : z_n)$ that vanishes on $\pi_p(\Gamma)$ and does not vanish on the line representing the point α. The polynomial P depends only on $(z_0 : z_2 : \cdots : z_n)$. and does not depend on z_1. Thus its zero locus is a union of fibers of the projection π_p containing $\pi_p^{-1}(\pi_p(\Gamma))$. Therefore, it vanishes on Γ and does not vanish at the point $q \in \pi_p^{-1}(\alpha)$. This proves the induction step. Chow Theorem is proved.

6 Biholomorphic automorphisms

In this series of lectures we will study biholomorphic automorphisms of some classical complex manifolds. First, we show that each automorphism of a projective space is projective. Then we present some results on automorphisms on bounded domains, with the complete classification of automorphisms of ball and polydisk. Their proofs are based on the Generalized Maximum Principle and Schwarz Lemma for holomorphic mappings of normed spaces.

6.1 Biholomorphic automorphisms of projective space

Theorem 6.1 *The group of biholomorphic automorphisms of projective space* $\mathbb{CP}^n$ *is the group* $PGL_{n+1}(\mathbb{C}) = PSL_{n+1}(\mathbb{C})$ *of projective transformations: each automorphism is a projective transformation.*

For the proof of Theorem 6.1 we introduce the notion of degree of a hypersurface (one-dimensional submanifold). We show that the intersection index of a hypersurface and a one-dimensional submanifold is equal to the product of their degrees. This implies that the images of intersecting line and hyperplane under a biholomorphism have degree one. Therefore, each biholomorphism of the projective space sends lines lo lines and hyperplanes to hyperplanes. This easily implies that it is a projective transformation.

Definition 6.2 The *degree* of a hypersurface $\Gamma \subset \mathbb{CP}^n$ is its intersection index with a line that is not contained in Γ. The *degree* of a one-dimensional connected complex submanifold in $\mathbb{CP}^n$ is its intersection index with a hyperplane that does not contains it. (The degrees under question do not depend on the choice of line (hyperplane), by Remark 5.16.)

Lemma 6.3 *The intersection index of a hypersurface and a one-dimensional connected complex submanifold in* $\mathbb{CP}^n$ *that is not contained in it is equal to the product of their degrees.*

Proof Let $\Gamma, \Lambda \subset \mathbb{CP}^n$ be respectively a hypersurface and a one-dimensional connected submanifold, $\Lambda \not\subset \Gamma$. Let d_1, d_2 denote their degrees. The intersection index is invariant under projective transformations in the following sense: for every $g_1, g_2 \in PGL_{n+1}(\mathbb{C})$ such that $g_2(\Lambda) \not\subset g_1(\Gamma)$ the intersection index $< g_1(\Gamma), g_2(\Lambda) >$ of the images is equal to the intersection index $< \Gamma, \Lambda >$, by connectedness of the group $PGL_{n+1}(\mathbb{C})$ and Remark 5.16. In more detail, given a one-dimensional connected complex manifold Λ and its image $g(\Lambda)$ under a projective transformation g, each of them being not contained in Γ, one can connect g to unity by a path g_t in $PGL_{n+1}(\mathbb{C})$, $g_0 = Id$, $g_1 = g$ such that $g_t(\Lambda) \not\subset \Gamma$ for all $t \in [0,1]$. It suffices to take an arbitrary point $p \in \Lambda$ with $p, g(p) \notin \Gamma$, connect p and $g(p)$ by path p_t in $\mathbb{CP}^n \setminus \Gamma$ and follow it by an arbitrary continuous family of projective transformations g_t such that $g_t(p) = p_t$, $g_0 = Id$, $g_1 = g$. Let us construct those transformations g_1 and g_2 for which the intersection index of the images is equal to $d_1 d_2$. To do this, fix a line $L \not\subset \Gamma$ that intersects Γ transversally at its regular points. Fix a projective hyperplane $\mathbb{CP}^{n-1}$ that intersects Λ transversally and does not contain L; then the latter intersections are finite. Fix a point $p \in L \setminus (\Gamma \cup \mathbb{CP}^{n-1})$ and a hyperplane $\mathbb{CP}^{n-2} \subset \mathbb{CP}^{n-1} \setminus \Lambda$. Consider the homogeneous coordinates $(z_0 : \cdots : z_n)$ on $\mathbb{CP}^n$ such that

$$p = (0 : 1 : \cdots : 0), \ \mathbb{CP}^{n-1} = \{z_1 = 0\}, \ \mathbb{CP}^{n-2} = \{z_0 = z_1 = 0\}$$

and the affine chart $\mathbb{C}^n = \{z_0 = 1\}$ equipped with the coordinates $z = (z_1, \dots, z_n) = (z_1, w)$, $w = (z_2, \dots, z_n)$, so that the line L be the z_1-axis. Recall that each affine transformation extends to a projective one. For every $0 < \lambda < 1$ consider the affine transformations

$$g_{1,\lambda}(z_1, w) = (\lambda z_1, w); \ g_{2,\lambda}(z_1, w) = (z_1, \lambda w).$$

Claim. *For every $\lambda > 0$ small enough one has $< g_{1,\lambda}(\Gamma), g_{2,\lambda}(\Lambda) >= d_1 d_2$.*

Proof Fix a polydisk $\Delta \subset \mathbb{C}^{n-1}$ centered at 0 in the w-space such that its preimage under the projection $\pi_1 : (z_1, w) \mapsto w$ intersects Γ by a union of d_1 graphs of holomorphic functions $z_1 = z_1(w)$ on Δ, see Fig. 2a): its existence follows from transversality and finiteness of the intersection $L \cap \Gamma$. Then the $g_{1,\lambda}$-images of the above graphs tend to the disk $\{0\} \times \Delta \subset \mathbb{C}^n$, as $\lambda \to 0$: the corresponding functions get multiplied by $\lambda \to 0$ and the multiplication results converge to zero uniformly on compact subsets in Δ, see Fig. 2b). Similarly, fix a disk D centered at 0 in the z_1-axis such that its preimage under the projection $\pi_2 : (z_1, w) \mapsto z_1$ intersects Λ by a union of d_2 graphs of holomorphic vector functions $w = w(z_1)$ on D. Its existence follows from transversality and the assumption that the points of intersection $\Lambda \cap \mathbb{CP}^{n-1}$ are finite points of the affine chart: the infinite hyperplane $\mathbb{CP}^{n-2} \subset \mathbb{CP}^{n-1}$ is disjoint from the curve Λ by assumption. Then the $g_{2,\lambda}$-images of the above graphs tend to $D \times \{0\}$, as $\lambda \to 0$, as above, see Fig. 2a),b). This implies that for every λ small enough the intersection index $< g_1(\Gamma), g_2(\Lambda) >$ equals the intersection index of d_1 copies of the polydisk $\{0\} \times \Delta$ and d_2 copies of the disk $D \times \{0\}$. That is, it is equal to $d_1 d_2$. The claim is proved. $\square$

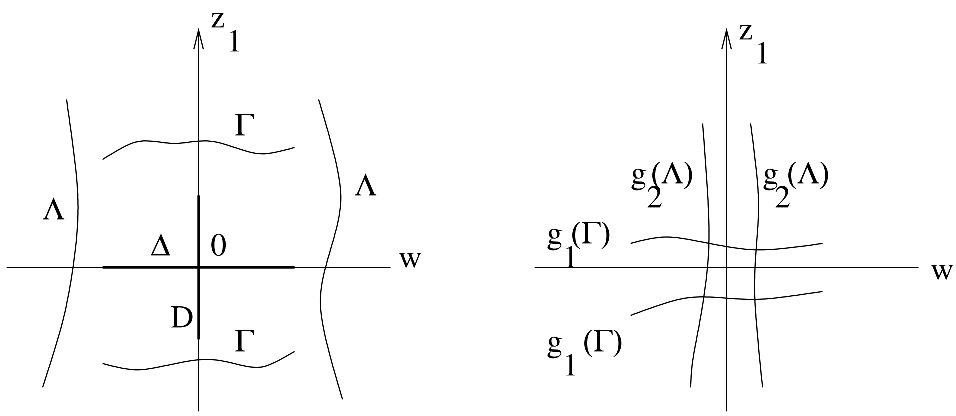

Figure 2: The hypersurface Γ, the curve Λ and their images under the mappings $g_1 = g_{1,\lambda}$ and $g_2 = g_{2,\lambda}$ respectively with small $\lambda > 0$.

The claim immediately implies Lemma 6.3, $\square$

Corollary 6.4 *Each hypersurface of degree one is a hyperplane. Each one-dimensional submanifold of degree one is a line.*

Proof Let a hypersurface $\Gamma \subset \mathbb{CP}^n$ be not a hyperplane. Then it is not contained in a hyperplane. Therefore, one can choose a line $L \not\subset \Gamma$ that intersects Γ in at least two distinct points. Therefore, Γ has degree at least two. The case of one-dimensional submanifold is treated analogously by changing line to a hyperplane that does not contain its connected component. The corollary is proved. $\square$

Proof of Theorem 6.1. Every biholomorphic mapping $g : \mathbb{CP}^n \to \mathbb{CP}^n$ sends hypersurface and line intersecting transversely to a hypersurface and a connected one-dimensional submanifold intersecting transversely at one point, thus having intersection index one. This together with Lemma 6.3 implies that the degrees of their images are equal to one. Therefore, the images are respectively a hyperplane and a line, by Corollary 6.4. Thus, g maps hyperplanes to hyperplanes and lines to lines. Without loss of generality we consider that it fixes a given hyperplane $\mathbb{CP}^{n-1}$: one can achieve this by replacing g by its post-composition with a projective transformation. Consider an affine chart on the complement $\mathbb{C}^n = \mathbb{CP}^n \setminus \mathbb{CP}^{n-1}$. Without loss of generality we can consider that g fixes the origin and the points represented by unit coordinate vectors: one can achieve this again by replacing g by its post-composition with a projective transformation that is an affine transformation in the chosen chart.

Claim. *A biholomorphic transformation $g : \mathbb{CP}^n \to \mathbb{CP}^n$ sending hyperplanes to hyperplanes and lines to lines, preserving an affine chart $\mathbb{C}^n$ and fixing the origin and unit coordinate vectors in the latter chart is the identity.*

Proof The transformation under question preserves the affine coordinate axes and fixes three points in each of them: the origin, the unity and the infinity. Therefore, it is the identity on each coordinate axis. It sends intersections of hyperplanes to intersections of hyperplanes. Therefore, it sends projective subspaces of any dimension to projective subspaces of the same dimension. Hence, it preserves the coordinates 2-planes. On each coordinate 2-plane E it is identity on both its coordinate axes, and hence, preserves each line $L \subset E$ that is not parallel to the axes: the intersection points of the line L with the axes are fixed. Hence, g preserves each line in E, and thus, the intersection point of every two lines is its fixed point. Finally, g is the identity on E. Analogously g is the identity on every coordinate subspace in $\mathbb{C}^n$ (induction in the dimension). The claim is proved. $\square$

The initial mapping g is a projective transformation, by construction and the claim. This proves Theorem 6.1. $\square$

6.2 Generalized Maximum Principle and Schwarz Lemma

We will be dealing with norms $|| \ ||$ on $\mathbb{C}^n$ positive on non-zero vectors and satisfying the following conditions:

$$||av|| + ||(1-a)v|| \leq ||v|| \text{ for every } a \in [0,1]; \tag{6.1}$$

$$||\lambda v|| = |\lambda| ||v|| \text{ for every } \lambda \in \mathbb{C}. \tag{6.2}$$

The unit ball centered at 0 in a given norm $|| \ ||$ will be denoted by

$$B_{|| \ ||} = \{||v|| < 1\}.$$

Condition (6.1) is equivalent to the convexity of the unit ball in the norm under consideration. Condition (6.2) is equivalent to its invariance under multiplication by complex numbers with unit module.

Example 6.5 The Euclidean norm and the maximum module norm

$$||z|| = ||z||_E = \sqrt{|z_1|^2 + \cdots + |z_n|^2}, \ ||z||_{\max} = \max_j |z_j|$$

satisfy conditions (6.1) and (6.2).

Theorem 6.6 (Generalized Maximum Principle). *Let $U \subset \mathbb{C}$ be a connected domain, $f : U \to \mathbb{C}^n$ be a holomorphic mapping. Let $|| \ ||$ be a norm on $\mathbb{C}^n$, and let the function $||f(z)||$ achieve its maximal value at some point $P \in U$. Then $||f(z)|| \equiv const$.*

In the proof of Theorem 6.6 we use the following general properties of convex sets and real hyperplanes in $\mathbb{C}^n$.

Theorem 6.7 *Let $C \subset \mathbb{R}^n$ be a convex subset. For every point $x \in \partial C$ there exists a hyperplane through x that does not intersect the interior $Int(C)$. Or equivalently, the interior of every convex subset is an intersection of half-spaces.*

Proof (sketch: a complete proof will be given as an exercise with hints in Task 4). It suffices to prove Theorem 6.7 for a bounded convex set: the intersection C_N of the set C with a ball centered at 0 of radius N. Namely, let H_N be hyperplanes through x that do not intersect $Int(C_N)$. Then we take H to be the limit of a converging subsequence H_{N_k} in the Grassmanian space of hyperplanes (which is compact).

Without loss of generality we consider that C is compact and $Int(C) \neq \emptyset$.

First we prove Theorem 6.7 in the case, when C is a polytope: the convex hull of a finite set. Afterwards we approximate C by polytopes and pass to the limit. Namely, for every $\varepsilon > 0$ fix a finite ε-net $S_\varepsilon \subset C$. Let $\Sigma_\varepsilon \subset C$ denote its convex hull, which is a polytope. For every $x_\varepsilon \to \partial\Sigma_\varepsilon$ there exists a hyperplane H_ε through x_ε satisfying the statement of the theorem for the convex set Σ_ε. Passing to the limit, as $\varepsilon \to 0$ and $x_\varepsilon \to x$, we take H to be the limit of a convergence subsequence H_{ε_k}. The hyperplane H passes through x and does not intersect $Int(C)$. This proves Theorem 6.7. $\square$

Proposition 6.8 *Every real hyperlane $H \subset \mathbb{C}^n$ (that is, every kernel of a $\mathbb{R}$-linear functional $\mathbb{C}^n \simeq \mathbb{R}^{2n} \to \mathbb{R}$) contains a complex subspace $H_{\mathbb{C}}$ of* *real codimension one in H. This is the intersection of the hyperplane H with its image under the multiplication by the imaginary unity i. It is the unique maximal complex subspace in H: every other complex subspace in H is contained in $H_{\mathbb{C}}$.*

Proof The latter multiplication being anti-involution ($i^2 = -1$), the image of the hyperplane H coincides with its preimage. This implies that the intersection $H_{\mathbb{C}} = H \cap (iH)$ is invariant under the multiplication by i, and hence, is a complex subspace: an additive subgroup invariant under multiplication by all complex numbers. Its codimension in H equals one. Indeed, otherwise, $H = iH$ would be a complex subspace in $\mathbb{C}^n$ of *real* codimension one, which is obviously impossible. Conversely, every complex subspace in H should be contained in $H_{\mathbb{C}}$, being invariant under the multiplication by i. The proposition is proved. □

Proof of Theorem 6.6. In the case, when $f(P) = 0$, the statement of the theorem is obvious. Thus, we consider that $f(P) \neq 0$. Fix a hyperplane H through the image $f(P)$ that does not intersect the ball $B = \{||w|| < ||f(P)||\}$: it exists by Theorem 6.7. Let $H_{\mathbb{C}} \subset H$ denote the maximal complex subspace. Let L denote the complex line $\mathbb{C}f(P)$ generated by the vector $f(P)$. The intersection $l = L \cap H$ is a real line in $L \simeq \mathbb{C}$. Consider the new affine coordinates $(w_1, \dots, w_m)$ in the image such that the line L be the w_1-axis, $P = 0$ and l be the imaginary axis in L in the coordinate w_1, and $H_{\mathbb{C}}$ be the coordinate $(w_2, \dots, w_m)$-hyperplane. In addition we require that the ball B lies on the side where $\operatorname{Re} w_1 \leq 0$: one can achieve this by changing the sing of the coordinate w_1, since $H \cap B = \emptyset$. Let $\pi_1 : \mathbb{C}^m \to \mathbb{C}$ denote the projection to the w_1-axis. One has $\operatorname{Re}(\pi_1 \circ f(z)) \leq 0$ for every $z \in U$, since $f(z) \in \overline{B} \subset \{\operatorname{Re} w_1 \leq 0\}$. Thus, we have that $g(z) = \pi_1 \circ f(z)$ is a holomorphic function on U, $g(P) = 0$, and $\operatorname{Re}(g(z)) \leq 0$. Therefore, $g(z) \equiv 0$ (Opennes Principle for holomorphic functions and uniqueness of analytic extension). Or equivalently, $f(z) \in H_{\mathbb{C}}$ for every $z \in U$. On the other hand, $||v|| \geq ||f(P)||$ for every $v \in H$, by the choice of the hyperplane H. Therefore, $||f(z)|| \geq ||f(P)||$ for every $z \in U$, and at the same time, $||f(P)||$ is the maximal value of the function $||f(z)||$. Hence, $||f(z)|| \equiv const$. Theorem 6.6 is proved. □

Lemma 6.9 (Generalized Schwarz Lemma). *Let $|| \ ||_1$, $|| \ ||_2$ be norms on $\mathbb{C}^n$ and $\mathbb{C}^m$ respectively. Let $f : B_{|| \ ||_1} \to B_{|| \ ||_2}$ be a holomorphic mapping such that $f(0) = 0$. Then $||f(z)||_2 \leq ||z||_1$.*

Proof Fix a complex line $L \subset \mathbb{C}^n$ through the origin. Its intersection with the unit ball $B_{|| \, ||_1}$ is a disk $D \subset L$ centered at zero. Let t be a linear complex coordinate on L in which D be the unit disk. Thus, the restriction to L of the norm $|| \, ||_1$ coincides with the function $|t|$. Consider the restriction of the mapping f to D as a holomorphic vector function in t. The function $g(t) = \frac{f(t)}{t}$ is holomorphic on the unit disk D, since $f(0) = 0$. There are two possible cases.

Case 1): $||g(t)||_2 \leq 1$ for every $t \in D$. This is equivalent to the inequality of the lemma for the restriction of the function f to D.

Case 2): $||g(t)||_2 > 1$ at some point t. The upper limit of the function $||g(t)||_2$, as $|t| \to 1$, is no greater than one, since $||f||_2 \leq 1$ on D, by assumption. Therefore, it takes its maximum greater than one at some point $t_0 \in D$. Hence, $||g(t)|| \equiv r > 1$ on D, by Theorem 6.6. That is, $||f(t)||_2 \equiv r|t|$, $r > 1$, and $||f(r^{-\frac{1}{2}})||_2 = r^{\frac{1}{2}} > 1$. This contradicts the condition of the lemma, which implies that $||f||_2$ takes values less than one on the disk D. Hence, this case is impossible. Lemma 6.9 is proved. □

6.3 Cauchy inequality. Henri Cartan's theorem on automorphisms tangent to identity

Definition 6.10 A complex manifold is called a *domain of bounded type,* if it is biholomorphic to a bounded domain in $\mathbb{C}^n$.

Here we prove the following theorem

Theorem 6.11 (Henri Cartan). *Let B be a domain of bounded type, $O \in B$, $f : B \to B$ be a biholomorphic automorphism such that $f(O) = O$ and $df(O) = Id$. Then $f = Id$.*

In the proof of Cartan's Theorem we use Cauchy inequality, which follows immediately from Cauchy Integral Formula.

Theorem 6.12 (Cauchy Inequality). *Let $f : \Delta_r \to \mathbb{C}$ be a holomorphic function on a polydisk of multiradius $r = (r_1, \dots, r_n)$, and let $|f| \leq R$ on Δ_r. Let $m \in (\mathbb{Z} \geq 0)^n$, and let c_m be the Taylor coefficient of the function f at 0 corresponding to the monomial z^m. Then*

$$|c_m| \leq \frac{R}{r^m}. \tag{6.3}$$

Proof Without loss of generality we consider that f is holomorphic on the closed polydisk $\overline{\Delta}_r$, replacing r by λr, $0 < \lambda < 1$ and passing to the limit, as $\lambda \to 1$. One has

$$c_m = (\frac{1}{2\pi i})^n \oint_{|\zeta_1|=r_1} \dots \oint_{|\zeta_n|=r_n} \frac{f(\zeta)}{\zeta^m \zeta_1 \dots \zeta_n} d\zeta_n \dots d\zeta_1. \tag{6.4}$$

Indeed, in the Laurent series of the function $\frac{f(z)}{z^m z_1 \dots z_n}$ each monomial different from $\frac{c_m}{z_1 \dots z_n}$ contains at least one coordinate z_j in a power different from -1. Hence, its integral over the product of boundaries ∂D_{r_j} vanishes, since the residue in the coordinate z_j vanishes. This implies that in the integral in the right-hand side of the formula (6.4) the only nontrivial contribution is given by the monomial $\frac{c_m}{\zeta_1 \dots \zeta_n}$, and the integral of the latter equals $(2\pi i)^n c_m$. This proves (6.4). The restriction to the product of the boundaries ∂D_{r_j} of the subintegral expression in (6.4) has module no greater than $\frac{R}{r^m r_1 \dots r_n}$. This together with (6.4) implies (6.3). □

Proof of Theorem 6.11. Without loss of generality we consider that $B \subset \mathbb{C}^n$ is a bounded domain, O is the origin and B contains the polydisk $\Delta = \Delta_{(1,1,\dots,1)}$. Let R denote the minimal radius of the ball centered at the origin that contains B. For every $m \in (\mathbb{Z}_{\geq 0})^n$ let c_m denote the coefficient at z^m in the Taylor series at 0 of the mapping f. Suppose the contrary: $f \neq Id$, that is, the Taylor series of the mapping f contains some nonlinear terms. Set

$$d = \min\{|m| = |m_1| + \dots + |m_n| \mid m \in (\mathbb{Z}_{\geq 0})^n, \ c_m \neq 0, \ |m| \geq 2\},$$

$$P_d(z) = \sum_{|m|=d} c_m z^m.$$

The polynomial $P_d(z)$ is homogeneous nonzero of degree d. Consider the iterations $f^k = f \circ f \circ \dots \circ f$. For every $k \in \mathbb{N}$ one has

$$f^k(z) = z + kP_d(z) + O(|z|^{d+1}) :$$

taking k-th iterate of a mapping tangent to the identity (i.e., fixing 0 and having identity derivative there) multiplies lower nonlinear terms by k. This follows immediately from the fact that the Taylor series of the composition of mappings is the formal composition of their Taylor series and straightforward calculation. Therefore, for every m with $|m| = d$ and $c_m \neq 0$ for every $k \in \mathbb{N}$ the coefficient at z^m of the k-th iterate f^k equals kc_m. Thus, it becomes arbitrarily large, as k is large enough. On the other hand, the latter coefficients kc_m should be no greater than R for all k, by Theorem 6.12 and since all the iterates f^k are holomorphic on Δ and take values in the ball of radius R centered at the origin. The contradiction thus obtained proves Theorem 6.11. □

6.4 Automorphisms of the ball and the polydisk

Here we will describe completely the above-mentioned automorphisms and prove the following theorem.

Theorem 6.13 *The ball and the polydisk are not biholomorphically equivalent.*

In the proof of Theorem 6.13 and in what follows we use the following simple observation.

Remark 6.14 The group $Aut(D_1)$ of conformal transformations of the unit disk acts transitively on it. The product $(Aut(D_1))^n$ acts transitively on the polydisk $\Delta = \Delta_{(1,\dots,1)}$.

Proof of Theorem 6.13. Suppose the contrary: there exists a biholomorphism $\Phi : B \to \Delta$ of the unit ball $B \subset \mathbb{C}^n$ onto the polydisk Δ. Without loss of generality we can and will consider that $\Phi(0) = 0$: one can achieve this by replacing Φ by its post-composition with a transformation from the group $(Aut(D_1))^n$, by transitivity, see the above remark. Thus, Φ is a biholomorphism of unit balls in the Euclidean norm $|| \, ||_E$ in the source and max-norm in the image that maps zero to zero. Therefore, $||\Phi(z)||_{\max} \equiv ||z||_E$, by Schwarz Lemma. Thus, for every $0 < r < 1$ a C^∞ diffeomorphism Φ sends the Euclidean sphere S_r of radius r onto the sphere Σ_r of radius r in the max-norm. Therefore, the sphere Σ_r is a C^∞-smooth submanifold, as is S_r. But this is false. Indeed, suppose that Σ_r is smooth (that is, regular) at the point $w = (r, \dots, r)$. Note that for every $j = 1, \dots, n$ the sphere Σ_r contains the product $D_1 \times \dots \times D_1 \times r \times D_1 \dots D_1$, where r stands at the j-th position. Therefore, for every j the tangent space $T_w\Sigma_r$ contains the sum of complex lines parallel to all the coordinate axes except for the j-th. Hence, $T_w\Sigma_r = T_w\mathbb{C}^n$, which is obviously impossible, if Σ_r is a submanifold of real codimension one, as is S_r. The contradiction thus obtained proves the theorem. □

Theorem 6.15 *The group of automorphisms of the unit polydisk $\Delta = \Delta_{(1,\dots,1)} \subset \mathbb{C}^n$ is generated by the product $(Aut(D_1))^n$ and the symmetric group S_n acting by permutations of coordinate components: each automorphism is a composition of an element of the above product and a permutation.*

Proof It suffices to prove the statement of the theorem for every automorphism $g \in Aut(\Delta)$ fixing the origin: each automorphism of the polydisk can be corrected to fix the origin by replacing it by its post-composition with an element of the group $(Aut(D_1))^n$, see Remark 6.14.

Proposition 6.16 *Let $g \in Aut(\Delta)$ fix 0. Then g is the composition of a permutation of coordinates and their multiplications by complex numbers with unit modules.*

Proof For every $j = 1, \dots, n$ let $V_j \subset \Delta$ denote the subset of those points $z = (z_1, \dots, z_n)$ for which $|z_j| > |z_s|$ for every $s \neq j$. The union $\cup_{j=1}^n V_j$ is an open and dense subset in Δ. One has $||z||_{\max} \equiv |z_j|$ on V_j, by definition. Let $k \in \{1, \dots, n\}$ be an index such that $V_k \cap g(V_j) \neq \emptyset$, or equivalently, $U_{jk} = V_j \cap g^{-1}(V_k) \neq \emptyset$. One has $||g(z)||_{\max} \equiv ||z||_{\max}$, by Schwarz Lemma. Therefore, $|z_k(g(z))| \equiv |z_j|$ on U_{jk}. Thus, the ratio of two holomorphic functions z_j and $z_k \circ g$ on the open set U_{jk} is holomorphic and has module identically equal to one. Therefore the latter ratio is locally constant, by Opennes Principle for holomorphic functions. Thus, there exists a $\theta \in \mathbb{R}$ such that $z_k \circ g \equiv e^{i\theta} z_j$ on an open subset in Δ, and hence, on all of Δ, by uniqueness of analytic extension. Finally, for every $j = 1, \dots, n$ there exist a $k = k(j)$ and a $\theta_j \in \mathbb{R}$ such that $z_{k(j)} \circ g \equiv e^{i\theta_j} z_j$. One has $k(j_1) \neq k(j_2)$ whenever $j_1 \neq j_2$, since g is invertible. Thus, the mapping $j \mapsto k(j)$ is a permutation. This proves the proposition. $\Box$

The proposition immediately implies the statement of Theorem 6.15. $\Box$

The next theorem describes the automorphisms of the unit ball B. To state it, let us consider the subgroup $U(1,n) \subset GL_{n+1}(\mathbb{C})$ acting naturally on the space $\mathbb{C}^{n+1}$ with the coordinates $\widetilde{z} = (\widetilde{z}_0, \dots, \widetilde{z}_n)$ that preserves the indefinite Hermitian form

$$Q(\widetilde{z}) = |\widetilde{z}_0|^2 - \sum_{j=1}^{n} |\widetilde{z}_j|^2.$$

Let $PU(1,n)$ denote its projectivization: its image under the natural projection $GL_{n+1}(\mathbb{C}) \to PGL_{n+1}(\mathbb{C})$ of factorization by $\mathbb{C}^*$. Set

$$K = \{Q > 0\} \subset \mathbb{C}^{n+1}, \ \Sigma = \{v \in \mathbb{C}^{n+1} \mid Q(v) = 1\} \subset K.$$

The images of the sets K and Σ under the tautological projection $\mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ coincide with the Euclidean unit ball B in the affine chart $\mathbb{C}^n = \{(1 : z_1 : \dots : z_n)\}$. The group $U(1,n)$ preserves both K and Σ. Therefore, each element of the group $PU(1,n)$ yields an automorphism of the unit ball.

Theorem 6.17 *The group of automorphisms of the unit ball $B \subset \mathbb{C}^n$ coincides with the group $PU(1,n)$: each its biholomorphism is the restriction to B of an element of the group $PU(1,n)$.*

The starting point of the proof of Theorem 6.17 is the following immediate corollary of Schwarz Lemma and Cartan's Theorem.

Lemma 6.18 *Every automorphism of the unit ball in $\mathbb{C}^n$ that fixes the origin is a unitary transformation.*

Proof Each automorphism $f(z)$ of the unit ball fixing the origin preserves the standard Euclidean norm: $||f(z)|| \equiv ||z||$, by Schwarz Lemma applied to f and to its inverse. Therefore, its differential $df(0)$ is a unitary operator. Without loss of generality we can and will consider that $df(0) = Id$: one can achieve this by taking a composition with appropriate unitary transformation. Then $f = Id$, by Cartan's Theorem. This proves the lemma. □

Remark 6.19 The group $U(n)$ of unitary transformations of the affine chart $\mathbb{C}^n$ embeds naturally into $PU(1,n)$. This follows from the fact that it lifts to the subgroup in $GL_{n+1}(\mathbb{C})$ fixing the $\widetilde{z}_0$-axis and acting as the unitary group $U(n)$ on the coordinates $(\widetilde{z}_1, \dots, \widetilde{z}_n)$.

Lemma 6.20 *The group $PU(1,n)$ acts transitively on the unit ball.*

Proof It suffices to show that $U(1,n)$ acts transitively on the unit sphere Σ in the pseudo-hermitian metric Q. That is, given two vectors $u, v \in \mathbb{C}^{n+1}$ with $Q(u) = Q(v) = 1$, let us show that there exists a transformation $g \in U(1,n)$ such that $g(u) = v$. Consider the orthogonal complements $u^\perp$ and $v^\perp$ with respect to the indefinite Hermitian form Q. One has $u \notin u^\perp$, $v \notin v^\perp$, since $Q(u) = Q(v) = 1 \neq 0$. The restriction to $u^\perp$ of the form Q is negative definite. Indeed, each indefinite Hermitian form has a well-defined signature: the number of positive squares minus the number of negative squares in a basis where its matrix is diagonal. The signature is independent on the choice of diagonalizing basis. The signature of the form Q is equal to $1 - n$. Its restriction to $u^\perp$ can be diagonalized: reduced to a sum of squared moduli of coordinates with signs. Then the signature of the form Q is equal to the signature of its restriction to $u^\perp$ plus one (corresponding to the vector u, where $Q(u) = 1 > 0$). This implies that the latter signature of restriction to $u^\perp$ equals $-n$, and thus, the latter restriction is negative definite. Finally, the restrictions of the form Q to both $u^\perp$ and $v^\perp$ are negative definite, and hence, can be transformed one into the other by a complex linear transformation $h : u^{\perp} \to v^{\perp}$. The transformation g sending u to v and coinciding with h on $u^{\perp}$ is a linear automorphism preserving the form Q, by construction, and hence, $g \in U(1,n)$. The lemma is proved. $\square$

The two latter lemmas immediately imply Theorem 6.17.

6.5 Introduction to complex dynamics: linearization theorem in dimension one

Here and in the next subsection we give an introduction to local complex dynamics given by a germ of biholomorphic transformation at a fixed point. We prove linearization theorems in one and two dimensions for contracting germs. Then we show that the attractive basin of an attracting non-resonant fixed point of an injective mapping $\mathbb{C}^2 \to \mathbb{C}^2$ is naturally biholomorphically equivalent to $\mathbb{C}^2$. This yields a wide class of domains in $\mathbb{C}^2$ that are smaller than $\mathbb{C}^2$ but biholomorphically equivalent to $\mathbb{C}^2$. This phenomena does not occur in one dimension, by Riemann Mapping Theorem.

Theorem 6.21 *Every germ of conformal mapping*

$$f : (\mathbb{C}, 0) \to (\mathbb{C}, 0),\ f(z) = \lambda z + O(z^2),\ 0 < |\lambda| < 1.$$

is conformally conjugated to its linear part. More precisely, there exists a unique germ $h : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$, $h(0) = 0$, $h'(0) = 1$, such that

$$\lambda h = h \circ f. \tag{6.5}$$

In the proof of Theorem 6.21 and its two-dimensional analogue we use the following classical theorem.

Theorem 6.22 *(Weierstrass). Let a family of functions holomorphic on a domain $D \subset \mathbb{C}^n$ converge uniformly on compact subsets in D. Then their limit is holomorphic.*

Proof The theorem in one variable is classical. Let us consider the case of higher dimensions. Let us consider a converging family as above. Then the limit is a continuous function. On the other hand, it is separately holomorphic, by the theorem in one variable. Hence, it is holomorphic, by Osgood's Lemma. $\square$

Proof of Theorem 6.21. Equation (6.5) is equivalent to the statement that h is a fixed point of the transformation

$$\mathcal{L} : h \mapsto \lambda^{-1} h \circ f.$$

We will show that $\mathcal{L}$ is a contraction in appropriate metric space and hence, has a unique fixed point there.

Fix a $\mu > 0$ such that

$$0 < \mu^2 < |\lambda| < \mu < 1. \tag{6.6}$$

Fix an $r > 0$ such that f is holomorphic on $\overline{D}_r$ and

$$\mu^2|z| \leq |f(z)| \leq \mu|z| \text{ whenever } z \in \overline{D}_r. \tag{6.7}$$

In particular, (6.7) implies that $f(\overline{D}_r) \subset D_r$.

For every function $q(z)$ holomorphic on D_r and continuous on $\overline{D}_r$ such that $q(0) = q'(0) = 0$ set

$$||q|| := \sup_{|z|\leq r} \frac{|q(z)|}{|z|^2}.$$

Let M denote the space of functions h holomorphic on D_r and continuous on $\overline{D}_r$ such that

$$h(0) = 0,\ h'(0) = 1,$$

equipped with the distance $dist(h_1, h_2) = ||h_1 - h_2||$. This is a complete metric space. Indeed, a sequence fundamental in the norm converges uniformly, by definition. Hence, its limit is holomorphic, by Weierstrass Theorem 6.22 and vanishes at 0. The derivatives also converge uniformly in compact set to the derivative of the limit, by Cauchy integral formula for the derivative and convergence of the function. Therefore, the limit has unit derivative at 0. Finally, the limit of a converging sequence is an element of the space M, and hence, M is complete.

Proposition 6.23 *$\mathcal{L}(M) \subset M$.*

Proof If $h(0) = 0$, then $(\mathcal{L}h)(0) = 0$ and $(\mathcal{L}h)'(0) = h'(0)$. If h is holomorphic on D_r and continuous on $\overline{D}_r$, then so is the composition $h \circ f$, since f is holomorphic on $\overline{D}_r$ and $f(\overline{D}_r) \subset D_r$. This implies that $\mathcal{L}$ preserves the space M and proves the proposition. □

Proposition 6.24 *$||\mathcal{L}h_1 - \mathcal{L}h_2|| \leq \nu||h_1 - h_2||$, $\nu = |\lambda|^{-1}\mu^2 < 1$.*

Proof The operator $\mathcal{L}$ being linear, it suffices to show that $||\mathcal{L}q|| \leq \nu||q||$ for every q as above. One has

$$\frac{|(\mathcal{L}q)(z)|}{|z^2|} = |\lambda|^{-1}\frac{|q(f(z))|}{|f(z)|^2}\frac{|f(z)|^2}{|z|^2} \leq |\lambda|^{-1}||q||\mu^2,$$

by definition, (6.7) and since $f(z) \in D_r$ whenever $z \in \overline{D}_r$. This implies that the norm of the image $\mathcal{L}q$ is no greater than $\nu||q||$. The proposition is proved. $\square$

The two latter propositions together imply that $\mathcal{L} : M \to M$ is a contraction. Hence, $\mathcal{L}$ has a unique fixed point $h \in M$, which obviously represents a conjugating germ we are looking for. Its uniqueness follows from the above uniqueness of fixed point and the fact that the above argument holds for every r small enough. This proves Theorem 6.21. $\square$

6.6 Linearization theorem in dimension two

Here we prove a linearization theorem for a germ $F = (f_1, f_2) : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ of biholomorphic mapping at 0 with linear part of the type

$$dF(0) = \Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \ 0 < |\lambda_1|, |\lambda_2| < 1.$$

Definition 6.25 A matrix Λ as above (or a vector $\lambda = (\lambda_1, \lambda_2)$) is said to be *resonant*, if it satisfies a relation of type

$$\lambda_j = \lambda^m = \lambda_1^{m_1}\lambda_2^{m_2}, \ m = (m_1, m_2) \in \mathbb{Z}^2_{\geq 0}, \ m_1 + m_2 \geq 2,$$

which is called a *resonance relation*. If there are no resonance relations, then Λ is called *non-resonant.*

Remark 6.26 If $0 < |\lambda_1|, |\lambda_2| < 1$, then each resonance relation (if any) takes the form $\lambda_1 = \lambda_2^k$, $k \in \mathbb{N}$ (up to permutation of indices), since in this case $|\lambda^m| < |\lambda_j|$, whenever $m_1 + m_2 \geq 2$ and $m_j > 0$.

Theorem 6.27 *Every germ F as above with non-resonant linear part is biholomorphically conjugated to its linear part. More precisely, there exists a unique biholomorphic germ $H : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$, $H(0) = 0$, $dH(0) = Id$ such that*

$$\Lambda H = H \circ F. \tag{6.8}$$

The proof of Theorem 6.27 is analogous to the above proof of Theorem 6.21. Equation (6.8) is equivalent to the statement that H is a fixed point of the linear operator

$$\mathcal{L} : H \mapsto \Lambda^{-1} H \circ F.$$

First we replace F by its conjugate whose lower nonlinear terms have high enough degree. Then we will show that $\mathcal{L}$ is a contraction in appropriate complete metric space, which will imply the existence and uniqueness of fixed point.

Proposition 6.28 *For every $N \in \mathbb{N}$ there exists a biholomorphic germ H_N : $(\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ of vector polynomial with components of degree at most N with $H(0) = 0$, $dH(0) = Id$ such that*

$$H_N \circ F \circ H_N^{-1}(z) = \Lambda z + O(|z|^N). \tag{6.9}$$

Proof Induction in N.

Induction base: $N = 2$, $H_N = Id$.

Induction step. Let the statement of the proposition be proved for $N = k$. Let us prove it for $N = k+1$. Let H_k be the germ given by the induction hypothesis for $N = k$. Then

$$F_k(z) = H_k \circ F \circ H_k^{-1}(z) = \Lambda z + P_k(z) + O(|z|^{k+1}),$$

$$P_k(z) = \begin{pmatrix} \sum_{s=0}^{k} a_s z_1^s z_2^{k-s} \\ \sum_{s=0}^{k} b_s z_1^s z_2^{k-s} \end{pmatrix}. \tag{6.10}$$

We show that there exists a germ of vector polynomial

$$h_k(z) = z + Q_k(z), \quad Q_k(z) = \begin{pmatrix} \sum_{s=0}^{k} \alpha_s z_1^s z_2^{k-s} \\ \sum_{s=0}^{k} \beta_s z_1^s z_2^{k-s} \end{pmatrix}$$

such that

$$h_k \circ F_k \circ h_k^{-1}(z) = \Lambda z + O(|z|^{k+1}). \tag{6.11}$$

Then $H_{k+1} = h_k H_k$ satisfies (6.9) with $N = k + 1$. This will prove the induction step and the proposition.

Homological equation on the coefficients of the vector polynomial Q_k.

One has

$$h_k \circ F_k \circ h_k^{-1}(z) = \Lambda z + P_k(z) + Q_k(\Lambda z) - \Lambda Q_k(z) + O(|z|^{k+1}).$$

Therefore, equation (6.11) is equivalent to the equation

$$P_k(z) + Q_k(\Lambda z) - \Lambda Q_k(z) = 0, \tag{6.12}$$

which is called the *homological equation.* The coefficient at $z_1^s z_2^{k-s}$ of the first (second) component in its left-hand side equals respectively

$$a_s + \alpha_s(\lambda_1^s \lambda_2^{k-s} - \lambda_1) = 0,$$

$$b_s + \beta_s(\lambda_1^s \lambda_2^{k-s} - \lambda_2) = 0.$$

Note that the above expressions in the brackets (the multipliers at α_s and β_s) are non-zero by non-resonance condition. Therefore, the latter equations in α_s and β_s can be solved, and the vector polynomial Q_k constructed from their solutions α_s, β_s satisfies (6.12), by construction. This proves the proposition. □

Proof of Theorem 6.27. Without loss of generality we consider that $|\lambda_1| \leq |\lambda_2|$. Fix a $\mu > 0$ such that

$$0 < |\lambda_1| \leq |\lambda_2| < \mu < 1. \tag{6.13}$$

Let us choose a $N \in \mathbb{N}$ large enough so that

$$|\lambda_1|^{-1}\mu^N < 1. \tag{6.14}$$

Without loss of generality we consider that

$$F(z) = \Lambda z + O(|z|^N).$$

One can achieve this by conjugation from the above proposition. We will be looking for a linearizing conjugation of the type $H(z) = z + O(|z|^N)$. Fix an $r > 0$, such that F is holomorphic on the closed Euclidean ball $\overline{B}_r$ of radius r and

$$|F(z)| \leq \mu|z| \text{ whenever } z \in \overline{B}_r. \tag{6.15}$$

Here the norm is Euclidean. Let M denote the space of holomorphic mappings $H : B_r \to \mathbb{C}^2$ continuous on $\overline{B}_r$ such that $H(0) = 0$, $H(z) = z + O(|z|^N)$. For every holomorphic mapping $Q : B_r \to \mathbb{C}^2$ continuous on $\overline{B}_r$ with

$$Q(z) = O(|z|^N) \text{ as } z \to 0$$

set

$$||Q|| = \sup_{z \in \overline{B}_r} \frac{|Q(z)|}{|z|^N}.$$

The space M equipped with the distance $d(H_1, H_2) = ||H_1 - H_2||$ is a complete metric space. The operator

$$\mathcal{L} : H \mapsto \Lambda^{-1} H \circ F$$

is a well-defined transformation of the space M to itself, since $F(\overline{B}_r) \subset B_r$, as in the previous subsection. Set

$$\nu = |\lambda_1|^{-1}\mu^N < 1.$$

Claim. *One has* $||\mathcal{L}Q|| \leq \nu||Q||$ *for every* Q *as above.*
Proof One has

$$\frac{|\Lambda^{-1}Q\circ F(z)|}{|z|^N} \leq |\lambda_1|^{-1}\frac{|Q\circ F(z)|}{|F(z)|^N}(\frac{|F(z)|}{|z|})^N \leq \lambda_1^{-1}\mu^N||Q|| = \nu||Q||,$$

as in the previous subsection. This implies the claim. □

The claim implies that $\mathcal{L}: M \to M$ is a contraction, and hence, it has a unique fixed point. This finishes the proof of Theorem 6.27, as at the end of the previous subsection. □

6.7 Polynomial automorphisms of $\mathbb{C}^2$. Fatou–Bieberbach domains

Here we study polynomial automorphisms of $\mathbb{C}^2$ having an attractive fixed point of non-resonant type. We show that its basin of attraction is biholomorphic to $\mathbb{C}^2$. In the case, when the basin is not all of $\mathbb{C}^2$ (e.g., if there is another fixed point), it yields an example of domain in $\mathbb{C}^2$ different from $\mathbb{C}^2$ but biholomorphic to $\mathbb{C}^2$: the so-called Fatou-Bieberbach domain. This phenomena is specific to higher dimensions and does not occur in dimension one: every simply connected domain in $\mathbb{C}$ different from all of $\mathbb{C}$ is conformally equivalent to the unit disk, not to $\mathbb{C}$ (Riemann Mapping Theorem).

Example 6.29 Here are some examples of biholomorphic automorphisms of $\mathbb{C}^2$:

1) The group of affine transformations generated by the group $GL_2(\mathbb{C})$ and the group $\mathbb{C}^2$ of translations.

2) Elementary polynomial automorphisms of higher degrees:

$$\Psi : \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \mapsto \begin{pmatrix} z_1 + P(z_2) \\ z_2 \end{pmatrix}.$$

3) Transcendental transformations, e.g., $(z_1, z_2) \mapsto (z_1 + e^{z_2}, z_2)$.

Theorem 6.30 *(***Jung, 1942***).*[1] *All the polynomial automorphisms, i.e., biholomorphisms of* $\mathbb{C}^2$ *given by vector polynomials form a group generated by affine and elementary polynomial automorphisms, see the above classes 1) and 2).*

[1] A beautiful geometric and relatively simple proof of Jung Theorem was obtained by a French mathematician Stéphane Lamy: Lamy, S. *Une prevue géométrique du théorème de Jung.* – Enseignement Mathématique, **48** (2002), 291–315.

We will not present a proof of Jung Theorem, since it requires additional techniques not covered by the cours.

Theorem 6.31 *Let $F : \mathbb{C}^2 \to \mathbb{C}^2$ be an injective holomorphic mapping that has a fixed point at the origin. Let its linear part $\Lambda = dF(0)$ be diagonal non-resonant with nonzero eigenvalues lying in the unit disk. Consider the attractive basin*

$$V = \{z \in \mathbb{C}^2 \mid F^k(z) \to 0, \text{ as } k \to +\infty\}.$$

Then the local linearizing germ $H : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ from Theorem 6.27 conjugating F to Λ (i.e., satisfying (6.8)) extends up to a biholomorphic isomorphism $H : V \simeq \mathbb{C}^2$.

Proof There exists a ball B centered at the origin such that H is well-defined and holomorphic on B and $F(\overline{B}) \subset B$ (see the proof of Theorem 6.27). Set

$$B_0 = B, \ B_1 = F^{-1}(B_0), \ B_2 = F^{-1}(B_1), \dots$$

One has

$$B_0 \subset B_1 \subset \dots = V,$$

since by definition, each point of the basin V is eventually sent to B by some iteration of the mapping F. We show that H extends holomorphically to every B_k by induction in k.

The induction base is obvious: H is holomorphic on B_0.

Induction step. Let we have already shown that H is holomorphic on B_k and satisfies (6.8) on B_k:

$$H = \Lambda^{-1} H \circ F. \tag{6.16}$$

Let us prove that it extends holomorphically to B_{k+1} and satisfies (6.16) there. The latter composition $\Lambda^{-1} H \circ F$ is well-defined and holomorphic on B_{k+1}, since $F(B_{k+1}) \subset B_k$, H is holomorphic on B_k (induction hypothesis) and Λ is invertible. It coincides with H on B_k (induction hypothesis: equality (6.16) on B_k). Therefore, it yields a holomorphic extension of the mapping H to B_{k+1}, and equation (6.16) holds on B_{k+1} by construction. The induction step is over. Theorem 6.31 is proved. □

Corollary 6.32 *Let $F : \mathbb{C}^2 \to \mathbb{C}^2$ be an injective holomorphic mapping (e.g., biholomorphic) that has a fixed point p whose linear part is diagonal non-resonant and has all the eigenvalues nonzero and lying in the unit disk. Then its attractive basin is biholomorphic to $\mathbb{C}^2$.*

Definition 6.33 A *Fatou–Bieberbach domain* is a domain in $\mathbb{C}^n$ different from $\mathbb{C}^n$ that is biholomorphically equivalent to $\mathbb{C}^n$. (These domains exist only for $n \geq 2$.)

Remark 6.34 In the case, when, e.g., F has an additional fixed point $q \neq p$, the attractive basin is different from all of $\mathbb{C}^2$, and hence, is a Fatou–Bieberbach domain.

Let us construct a polynomial automorphism with an attractive basin being a Fatou–Bieberbach domain. Take polynomial automorphisms

$$f : (z_1, z_2) \mapsto (z_1 + z_2, z_2); \;\; g : (z_1, z_2) \mapsto (z_1, z_2 + z_1^2).$$

Let us choose a non-resonant diagonal matrix

$$\Lambda = \operatorname{diag}(\lambda_1, \lambda_2), \;\; \lambda_1 \neq \lambda_2, \;\; 0 < |\lambda_1|, |\lambda_2| < 1.$$

Set

$$F(z) = \Lambda g \circ f(z) = \begin{pmatrix} \lambda_1(z_1 + z_2) \\ \lambda_2(z_2 + (z_1 + z_2)^2) \end{pmatrix}.$$

Proposition 6.35 *The attractive basin V of the fixed point 0 of the automorphism F is biholomorphically equivalent to $\mathbb{C}^2$. The automorphism F has an additional fixed point $q \neq 0$, hence $V \neq \mathbb{C}^2$ is a Fatou–Bieberbach domain.*

Proof The differential $dF(0)$ has distinct eigenvalues λ_1, λ_2, and hence, is conjugated to the diagonal matrix. Therefore, F is linearizable on V (Theorems 6.27 and 6.31). The system of equations on fixed points has the form

$$\begin{cases} z_1 = \lambda_1(z_1 + z_2) \\ z_2 = \lambda_2(z_2 + (z_1 + z_2)^2) \end{cases} \tag{6.17}$$

The first equation of the system is equivalent to each one of the two following equations:

$$z_1 = \frac{\lambda_1 z_2}{1 - \lambda_1}, \;\; z_1 + z_2 = z_2(1 + \frac{\lambda_1}{1 - \lambda_1}) = \frac{z_2}{1 - \lambda_1}.$$

Substituting the latter expression for $z_1 + z_2$ to the second equation in (6.17) and dividing it by z_2 yields

$$1 + \frac{z_2}{(1 - \lambda_1)^2} = \lambda_2^{-1}.$$

This yield a solution

$$z_2 = (1-\lambda_1)^2(\lambda_2^{-1}-1), \ z_1 = \frac{\lambda_1 z_2}{1-\lambda_1} = \frac{\lambda_1}{\lambda_2}(1-\lambda_1)(1-\lambda_2)$$

of system (6.17), and hence, an additional fixed point of the mapping F. The proposition is proved. $\square$

7 Domains of holomorphy. Holomorphic convexity. Pseudoconvexity. Riemann domains.

Here we introduce the notion of domain of holomorphy: a domain that admits a holomorphic function "everywhere non-extendable" through the boundary. We prove Oka's Theorem, which says that being domain of holomorphy is equivalent to holomorphic convexity. Then we study local versions of convexity: *pseudo-convexity, Levi convexity,...* which appear to be equivalent to the global holomorphic convexity.

In the present section for every $r > 0$ the polydisk centered at z_0 with multiradius $(r, \dots, r)$ will be denoted by

$$\Delta(z_0, r) = \Delta_{r,\dots,r}(z_0) = (D_r(z_0))^n.$$

7.1 Domains of holomorphy and holomorphic convexity. Oka's Theorem

Let $D \subset \mathbb{C}^n$. For every $z_0 \in D$ set

$$r(z_0) = \max\{r > 0 \mid \Delta(z_0, r) \subset D\}$$

Definition 7.1 A domain $D \subset \mathbb{C}^n$ is called a *domain of holomorphy*, if there exists a holomorphic function $f : D \to \mathbb{C}$ such that for every $z_0 \in D$ the function $f|_{\Delta(z_0, r(z_0))}$ cannot be extended holomorphically to a bigger polydisk $\Delta(z_0, R)$, $R > r(z_0)$.

Example 7.2 The unit disk $D_1 \subset \mathbb{C}$ is a domain of holomorphy. For example, the modular function $f : D_1 \to \mathbb{C}$ (providing the universal covering over $\overline{\mathbb{C}} \setminus \{0, 1, \infty\}$ and obtained by reflecting ideal hyperbolic triangles) does not extend in the above sense, since it takes values arbitrarily close to 0, 1, ∞ in a neighborhood of every point of the boundary. Similarly, every simply connected domain $V \subset \mathbb{C}$ is a domain of holomorphy with respect to the composition of a conformal isomorphism $V \simeq D_1$ and the modular function $D_1 \to \mathbb{C}$. Moreover, one can show that *every domain in* $\mathbb{C}$ *is a domain of holomorphy.*

Everywhere below for a domain $D \subset \mathbb{C}^n$ by $H(D)$ we denote the space of holomorphic functions on D.

The following definition generalizes the notion of convexity.

Definition 7.3 Let $D \subset \mathbb{C}^n$ be a domain, $K \subset D$ be a subset. Fix a class of functions $F \subset H(D)$. Let us define the *F-convex hull*

$$\hat{K}_F = \{z \in D \mid |f(z)| \leq \sup_{x \in K} |f(x)| \text{ for every } f \in F\}.$$

The subset K is called *F-convex,* if $\hat{K}_F = K_F$ (then it is automatically closed). The *domain D is called F-convex,* if the F-hull $\hat{K}$ of every compact subset $K \Subset D$ is compact. In the case, when $F = H(D)$ we call the above F-convex objects *holomorphically convex* and denote $\hat{K} = \hat{K}_{H(D)}$.

Remark 7.4 Recall that the convex hull of a subset $K \subset \mathbb{R}^n$ is an intersection of half-spaces, each of them being bounded by a hyperplane through some point of the set K. Therefore, the convex hull can be defined as the set of those points $x \in \mathbb{R}^n$ such that for every linear functional l on $\mathbb{R}^n$ one has $l(x) \leq \sup_{y \in K} l(y)$. In the case, when $K \subset \mathbb{C}^n$, each half-space is defined by the inequality $\mathrm{Re}\, l < c$, where l is a $\mathbb{C}$-linear functional on $\mathbb{C}^n$, as in the proof of Theorem 6.6. Or equivalently, by inequality $|f_l| < e^c$, $f_l(z) = e^{l(z)}$. This implies that the convex hull of the set K coincides with its F-hull in $\mathbb{C}^n$ with respect to the class F consisting of the exponents of the $\mathbb{C}$-linear functionals.

Example 7.5 A ball centered at the origin is F-convex for F being the collection of $\mathbb{C}$-linear functionals.

Remark 7.6 Let $D \subset \mathbb{C}^n$, $F_1 \subset F_2 \subset H(D)$. Then every F_1-convex subset $K \subset D$ is always F_2- and $H(D)$-convex. One has $\hat{K}_{F_1} \supset \hat{K}_{F_2} \supset \hat{K} \supset K$. Similarly, if D is F_1-convex, then it is F_2-and $H(D)$-convex.

Remark 7.7 The $H(D)$-hull of a bounded subset K is bounded, since the modules of the coordinate functions cannot achieve values on $\hat{K}$ greater than their suprema on K.

One of the main results in the theory is the following theorem.

Theorem 7.8 (Oka). *A domain $D \subset \mathbb{C}^n$ is a domain of holomorphy, if and only if it is holomorphically convex.*

Corollary 7.9 *The notion of domain of holomorphy is invariant under biholomorphisms, as is the notion of holomorphic convexity.*

First we prove Oka's Theorem. Afterwards we show that convergence domains of power series are always holomorphically convex and characterize them.

The first step for Oka's Theorem is the next theorem.

Theorem 7.10 *Let a domain D be holomorphically convex. Then it is a domain of holomorphy.*

Proof In the proof of Theorem 7.10 we use the following proposition.

Proposition 7.11 *Let $D \subset \mathbb{C}^n$ be an F- convex domain. Then it admits a compact F-convex exhaustion*

$$K_1 \Subset K_2 \Subset \cdots = D, \; \hat{K}_j = K_j.$$

Proof Consider an arbitrary compact exhaustion $B_1 \Subset B_2 \Subset B_3 \Subset \cdots = D$. Set

$$j_1 = 1, \; K_1 = \hat{B}_{1,F} = \widehat{(B_1)}_F, \; j_2 = \min\{j \mid B_j \Supset K_1\}, \; K_2 = \hat{B}_{j_2,F}, \ldots .$$

The sets $K_1 \Subset K_2 \Subset \ldots$ form a compact F-convex exhaustion of the domain D. The proposition is proved. $\square$

Fix an $H(D)$-convex exhaustion $K_1 \Subset K_2 \Subset \ldots$ and a sequence of points $w_j \in K_{j+1} \setminus K_j$ accumulating to the boundary ∂D so that each open set intersecting the boundary ∂D contains a limit point of the sequence w_j: one can construct the latter sequence w_j, since K_j form a compact exhaustion of the domain D. We will construct a function $f \in H(D)$ such that $f(w_j) \to \infty$, as $j \to \infty$. This will imply that f is non-extendable to polydisks $\Delta(z_0, R)$, $z_0 \in D$, $R > r(z_0)$: the latter polydisk intersects the boundary, and hence, contains a limit point of the sequence w_j; thus, f cannot extend holomorphically there. To do this, we construct functions $f_j \in H(D)$, set

$$F_k = \sum_{j=1}^{k} f_j,$$

with the following properties:

$$|f_j||_{K_j} < 2^{-j}, \ |F_j(w_j)| > 2^j. \tag{7.1}$$

and prove the above statements for the function

$$f = \sum_{j=1}^{+\infty} f_j.$$

We construct the functions f_j inductively, taking $f_0 = 0$ as the induction base. Let we have already constructed f_j for $j \leq l-1$. Let us construct f_l. The compact K_l is holomorphically convex, and $w_l \in K_{l+1} \setminus K_l$. This implies that there exists a holomorphic function $g : D \to \mathbb{C}$ such that

$$g(w_l) = 1, |g||_{K_l} < \delta < 1.$$

Set

$$f_l = (\frac{g}{\sqrt{\delta}})^N,$$

where N is chosen large enough so that

$$\delta^{\frac{N}{2}} < 2^{-l}, \ |f_l(w_l)| = \delta^{-\frac{N}{2}} > 2^l + |F_{l-1}(w_l)|.$$

The first inequality implies that $|f_l||_{K_l} < 2^{-l}$. The second one implies that

$$|F_l(w_l)| \geq |f_l(w_l)| - |F_{l-1}(w_l)| > 2^l.$$

The induction step is over. The functions f_j satisfying (7.1) are constructed. The first inequality in (7.1) together with Weierstrass Theorem imply that the series $f = \sum_{j=1}^{+\infty} f_j$ converges uniformly on compact subsets in D, and the limit f is holomorphic on D. For every $l \in \mathbb{N}$ one has

$$F_l(w_l) > 2^l, \ f_j(w_l) < 2^{-j} \text{ for every } j \geq l+1.$$

The first inequality follows from (7.1). The second one follows from the first inequality in (7.1) and the inclusion $w_l \in K_{l+1}$. Therefore

$$|f(w_l)| \geq |F_l(w_l)| - \sum_{j \geq l+1} |f_j(w_l)| \geq 2^{l-1}, \ f(w_l) \to \infty, \text{ as } l \to \infty.$$

This proves the theorem. □

Theorem 7.12 *Let $D \subset \mathbb{C}^n$ be a domain of holomorphy. Then it is holomorphically convex.*

For the proof of Theorem 7.12 we use the following notation and theorem. For every subset $K \subset D$ set

$$\rho(K, \partial D) = \inf\{r(z_0) \mid z_0 \in K\}.$$

Theorem 7.13 (Cartan–Thullen). *Let $D \subset \mathbb{C}^n$, $K \Subset D$ be a compact subset, $\sigma = \rho(K, \partial D)$. Then for every $z_0 \in \hat{K}$ every function $f \in H(D)$ extends holomorphically from z_0 to the polydisk $\Delta(z_0, \sigma)$.*

Proof To show that the function f is holomorphic on $\Delta(z_0, \sigma)$, we show that its Taylor series converges uniformly on compact subsets in $\Delta(z_0, \sigma)$. To do this, we estimate its Taylor coefficients at z_0 and by using Cauchy Inequality. Fix a $0 < \delta < \sigma$ and a function $f \in H(D)$. For every point $z_0 \in D$ and every $k \in \mathbb{Z}^n_{\geq 0}$ let $c_k(z_0)$ denote the Taylor coefficient at $(z-z_0)^k$ of the function f at z_0. One has

$$|\frac{\partial^{|k|}}{\partial z^k} f(z_0)| \leq \sup_K |\frac{\partial^{|k|}}{\partial z^k} f|,$$

since $z_0 \in \hat{K}$ and by the $H(D)$-convex hull inequality applied to the above partial derivative instead of the function f. This implies that

$$|c_k(z_0)| \leq \sup_{t \in K} |c_k(t)|, \tag{7.2}$$

since the Taylor coefficients are equal to the corresponding derivatives divided by the factorials of the components of the vector k. For every $t \in K$ one has $r(t) \geq \sigma > \delta$, by definition. Therefore, $K^\delta = \overline{\cup_{t \in K} \Delta(t, \delta)} \subset D$ is a compact subset. Set $M = \sup_{K^\delta} |f|$. One has

$$|c_k(t)| \leq \frac{M}{\delta^{|k|}} \text{ for every } t \in K,$$

by Cauchy Inequality. Hence, $c_k(z_0) \leq \frac{M}{\delta^{|k|}}$, by (7.2). Now fix an arbitrary $0 < \mu < \delta$. The series $\sum |c_k(z_0)|\mu^{|k|}$ converges. Indeed, its terms are no greater than $M\nu^{|k|}$, $\nu = \frac{\mu}{\delta} < 1$. The series $\sum_k \nu^{|k|} = (\frac{1}{1-\nu})^n$ converges. This together with the previous majoration implies that the series $\sum c_k(z_0)(z - z_0)^k$ converges uniformly on the polydisk $\Delta(z_0, \mu)$, where μ can be taken arbitrarily close to δ. On the other hand δ can be taken arbitrarily close to σ. Finally, the latter series, which is the Taylor series of the function f at

z_0 converges uniformly on compact subsets in the polydisk $\Delta(z_0, \sigma)$. Hence, f extends holomorphically there. The theorem is proved. $\square$

Proof of Theorem 7.12. Let D be a domain of holomorphy of a function f. Let $K \Subset D$ be an arbitrary compact set, $\sigma = \rho(K, \partial D)$. For every $z_0 \in \hat{K}$ the function f extends holomorphically to $\Delta(z_0, \sigma)$, by Theorem 7.13. This implies that $\sigma \leq r(z_0)$, by the definition of domain of holomorphy. Or equivalently,

$$r(z_0) \geq \rho(K, \partial D) \text{ for every } z_0 \in \hat{K}. \tag{7.3}$$

Finally, the gap between the subset $\hat{K} \subset D$ and ∂D is bounded from below by $\rho(K, \partial D)$, and $\hat{K}$ is bounded, see Remark 7.7. Hence, $\hat{K}$ is compact. Theorem 7.12 is proved. $\square$

Proof of Theorem 7.8. Theorem 7.8 follows immediately from Theorems 7.10 and 7.12. $\square$

Corollary 7.14 *Let $D \subset \mathbb{C}^n$ be a domain of holomorphy (or equivalently, holomorphically convex). Then for every compact subset $K \Subset D$ one has*

$$\rho(\hat{K}, \partial D) = \rho(K, \partial D).$$

Proof The corollary follows immediately from inequality (7.3) and the obvious inequality $\rho(\hat{K}, \partial D) \leq \rho(K, \partial D)$, which follows from the inclusion $K \subset \hat{K}$. $\square$

7.2 Characterization of domains of convergence of power series

Fix affine coordinates on $\mathbb{C}^n$ centered at 0. Consider the action on $\mathbb{C}^n$ of the n-torus $\mathbb{T}^n = (S^1)^n$, $S^1 = \mathbb{R}/2\pi\mathbb{Z}$, centered at 0:

$$\theta = (\theta_1, \dots, \theta_n) : z \mapsto (e^{i\theta_1} z_1, \dots, e^{i\theta_n} z_n).$$

Similarly we define the $\mathbb{T}^n$-action centered at arbitrary point $p \in \mathbb{C}^n$: the action is defined in the same way but in the affine coordinates with the origin shifted to p:

$$z \mapsto (p_1 + e^{i\theta_1}(z_1 - p_1), \dots, p_n + e^{i\theta_n}(z_n - p_n)).$$

We will consider the logarithmic mapping

$$\lambda : (\mathbb{C}^*)^n \to \mathbb{R}^n, \ z \mapsto (\ln |z_1|, \dots, \ln |z_n|).$$

Definition 7.15 A *Reinhardt domain* in $\mathbb{C}^n$ centered at a point $p \in \mathbb{C}^n$ is a domain invariant under the $\mathbb{T}^n$-action centered at p. A Reinhardt domain centered at p is *complete,* if it is a union of polydisks centered at p.

Definition 7.16 A $\mathbb{T}^n$-invariant subset $D \subset \mathbb{C}^n$ is *logarithmically convex,* if the image $\lambda(D \cap (\mathbb{C}^*)^n) \subset \mathbb{R}^n$ is convex.

Example 7.17 Every polydisk in $\mathbb{C}^n$ centered at the origin is logarithmically convex. Its image under the mapping λ is a negative quadrant $\{x_1 < a_1, \dots, x_n < a_n\}$, $a_1, \dots, a_n \in \mathbb{R}$. A Reinhardt domain is complete, if and only if its λ-image is a union of negative quadrants.

Example 7.18 For every power series $\sum_k c_k z^k$ its convergence domain is a complete logarithmically convex Reinhardt domain, by Corollary 2.4 and an exercise in Task 1. The next theorem provides the converse statement.

Theorem 7.19 *A domain $D \subset \mathbb{C}^n$ is the convergence domain of some power series $\sum_k c_k z^k$, if and only if it is a complete logarithmically convex Reinhardt domain.*

For the proof of the theorem we have to prove that each logarithmically convex complete Reinhardt domain is a convergence domain. To do this, we first prove that every *holomorphically* convex complete Reinhardt domain is a convergence domain. Then we show that each logarithmically convex complete Reinhards domain is holomorphically convex. This will prove the theorem.

Proposition 7.20 *Let $D \subset \mathbb{C}^n$ be a holomorphically convex complete Reinhardt domain. Then it is the convergence domain for some power series $\sum_k c_k z^k$.*

Proof The domain D under consideration is a domain of holomorphy, by Theorem 7.10. Therefore, there exists a holomorphic function $f : D \to \mathbb{C}$ that does not extend holomorphically beyond the boundary ∂D. This implies that its Taylor series at 0 converges on every polydisk $\Delta_r \subset D$, and does not converge on any bigger polydisk centered at 0. Hence, it converges on D, which is the union of the latter polydisks Δ_r, and thus, D is contained in its convergence domain. The convergence domain is a union of polydisks, and all of them are contained in D, by the above statement of non-convergence in bigger polydisks. Therefore, the domains under question coincide. The proposition is proved. □

Lemma 7.21 *Every logarithmically convex complete Reinhardt domain is holomorphically convex.*

Proof For the proof of the lemma we consider the completed real line $\hat{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$. The mapping λ is thus extended continuously as a mapping $\lambda : \mathbb{C}^n \to (\hat{\mathbb{R}})^n$.

Let D be a logarithmically convex complete Reinhardt domain. Let $K \Subset D$ be a compact subset containing a polydisk Δ_σ centered at the origin. Set

$$\hat{K}_{\log} = \lambda^{-1}(conv(\lambda(K)) :$$

this is the logarithmically convex hull of the set K. It is logarithmically convex, by definition. Note that the subset $\lambda(K) \subset (\hat{\mathbb{R}})^n$ is compact, as is K, since the mapping λ is continuous. Therefore, its convex hull is closed. It is also a bounded subset in $(\hat{\mathbb{R}})^n$. That is, the coordinate functions x_j on $(\hat{\mathbb{R}})^n$ are bounded from above on $conv(\lambda K)$, being bounded from above on $\lambda(K)$, as are $|z_j| = e^{x_j \circ \lambda}$. Therefore, the logarithmic hull $\hat{K}_{\log} \subset \mathbb{C}^n$ is a bounded closed, hence compact subset. One has $\hat{K}_{\log} \subset D$: $K \subset D$, hence $\lambda(K) \subset \lambda(D)$ and $conv(\lambda K) \subset \lambda(D)$, since $\lambda(D)$ is convex by definition. Set

$\mathcal{M} = \{z^k \mid k = (k_1, \dots, k_n) \in \mathbb{Z}^n_{\geq 0}\}$: this is the class of all the monomials.

Proposition 7.22 *Every logarithmically convex compact subset in $\mathbb{C}^n$ containing a polydisk centered at the origin is $\mathcal{M}$-convex.*

Proof Let $K \Subset D$ be a logarithmically compact subset containing a polydisk centered at the origin. Fix an arbitrary $w \in \mathbb{C}^n \setminus K$. It suffices to show that there exists a $k \in (\mathbb{Z}_{\geq 0})^n$ such that

$$|z^k(w)| > \sup_K |z^k|. \tag{7.4}$$

One has $\lambda(w) \notin \lambda(K)$. The set $\lambda(K)$ is convex, compact and contains a negative quadrant $\{x_1 \leq c_1, \dots, x_n \leq c_n\}$. Therefore, there exists a hyperplane L through $\lambda(w)$ disjoint from $\lambda(K)$, and thus, it does not intersect the latter quadrant. Let $\mathcal{N}$ denote the unit normal vector to L that is directed to the side separated from the latter quadrant by L. The vector $\mathcal{N} = (\mathcal{N}_1, \dots, \mathcal{N}_n)$ has all the components non-negative, by the latter direction condition. One has $\sum_j \mathcal{N}_j x_j(\lambda(w)) > \max_{x \in \lambda(K)}(N, x)$, by definition. Or equivalently,

$$|z^{\mathcal{N}}|(w) > \max_K |z^{\mathcal{N}}|, \ |z^{\mathcal{N}}| = \prod_{j=1}^n |z_j|^{\mathcal{N}_j}. \tag{7.5}$$

Let $\varepsilon > 0$ denote difference of the left-hand side and right-hand side of the latter inequality. We can approximate the positive components of the vector N by rationals: if the approximation is good enough, then the same inequality remains valid. Moreover we can chose the approximants having the same denominator q: $q\mathcal{N} \in (\mathbb{Z}_{\geq 0})^n$. Thus, without loss of generality we consider that $k = q\mathcal{N} \in (\mathbb{Z}_{\geq 0})^n$. Then inequality (7.5) holds by construction: taking q-th powers of the sides of inequality (7.5) keeps the inequality valid. This proves the proposition. $\square$

The logarithmic hull $\hat{K}_{\log}$ is $\mathcal{M}$-convex, by the proposition. Hence, it is $H(D)$-convex, since $\mathcal{M} \subset H(D)$. This proves the lemma. $\square$

Proof of Theorem 7.19. Each convergence domain of power series is a complete logarithmically convex Reinhardt domain (Corollary 2.4 and an exercise in Task 1). The converse follows from Proposition 7.20 and Lemma 7.21. Theorem 7.19 is proved. $\square$

7.3 Continuity Principle. Levi convexity

Definition 7.23 A domain $D \subset \mathbb{C}^n$ is *not holomorphically extendable* at a point $\zeta \in \partial D$, if there exist a neighborhood $U = U(\zeta) \subset \mathbb{C}^n$ and a holomorphic function $f : U \cap D \to \mathbb{C}$ that does not extend holomorphically to ζ. We say that D is *holomorphically non-extendable,* if it is not holomorphically extendable at each point of its boundary.

Remark 7.24 A domain of holomorphy is obviously holomorphically non-extendable. The converse statement was a problem stated by Levy and solved by Oka.

Theorem 7.25 (Oka). *A domain in $\mathbb{C}^n$ is a domain of holomorphy, if and only if it is holomorphically non-extendable.*

Example 7.26 Consider the Hartogs' figure H: the union of two subsets $A, B, \subset \mathbb{C}^2$,

$$A = \{r < |z_1| < R\} \times D_R,\ B = D_R \times V,\ V \subset D_R \text{ is an open subset.}$$

It is holomorphically extendable at the points of the product $D_R \times \partial V$. The reason is that for every point $w_0 \in \partial V$, set $r < \sigma < R$, the closed disk $S = \overline{D}_\sigma \times \{w_0\}$, which not contained in H, has the following properties:

- its boundary is contained in H;

- it is the limit of the family of disks $S_w = \overline{D}_\sigma \times \{w\}$, $w \in V$, contained in H.

The two latter statements together imply that the Cauchy integrals along their boundaries extend every holomorphic function to the limit disk S.

The next theorem states that presence of limiting embedded "disks" satisfying the two latter statements is basically the only reason for holomorphic extendability of a domain to its boundary point. To state it, let us introduce the following notions.

Definition 7.27 Let $n > r \geq 1$. Let $W \subset \mathbb{C}^r$ be a domain with compact closure, $\phi : \overline{W} \to \mathbb{C}^n$ be an injective holomorphic mapping, whose differential has maximal rank r at each point. The image

$$S = \phi(\overline{W})$$

is called a *compact holomorphic surface.*

Recall that for every subset $K \subset \mathbb{C}^n$ and each $r > 0$ we set

$$K^\delta = \cup_{t \in K} \Delta(t, \delta).$$

Definition 7.28 A sequence of subsets $M_k \subset \mathbb{C}^n$ *converges* to a subset $M \subset \mathbb{C}^n$, if for every $\varepsilon > 0$ there exists a $N > 0$ such that for every $k > N$ one has

$$M_k \subset M^\varepsilon \text{ and } M \subset M_k^\varepsilon.$$

Remark 7.29 The limit set is always closed.

Theorem 7.30 (Benke–Sommer Continuity Principle). *Let $D \subset \mathbb{C}^n$, $S_k \subset D$ be a sequence of compact holomorphic surfaces converging to a subset $S \subset \mathbb{C}^n$ whose boundaries ∂S_k converge to a subset $\Gamma \Subset D$. Then every holomorphic function $f : D \to \mathbb{C}$ extends holomorphically to a neighborhood of the limit set S.*

Proof Let us choose an auxiliary open subset G, $\Gamma \Subset G \Subset D$: the closure $\overline{G}$ is a compact subset in D and Γ is a compact subset in G. Set

$$r = \rho(G, \partial D).$$

There exists a $N > 1$ such that for every $k > N$ one has $\partial S_k \subset G$. Therefore, for those k every holomorphic function $f : D \to \mathbb{C}$ satisfies the inequality

$$\sup_{S_k} |f(z)| = \sup_{\partial S_k} |f(z)| \leq \sup_G |f(z)|.$$

The first equality follows from the Maximum Principle applied to the restriction of the function f to the surface S_k. The latter inequality implies that $S_k \subset \hat{G} = \hat{G}_{H(D)}$ for every $k > N$. Therefore, for those k f extends holomorphically to the neighborhood S_k^r (Cartan–Thullen Theorem 7.13.) For every k large enough one has $S \subset S_k^{\frac{r}{2}}$, hence $S^{\frac{r}{2}} \subset S_k^r$, by convergence. This implies that f extends holomorphically to $S^{\frac{r}{2}}$. This proves the theorem. □

Remark 7.31 One can show (slightly modifying the above proof) that f extends holomorphically to S^ρ, $\rho = \rho(\Gamma, \partial D)$.

As an application of the Continuity Principle, let us prove the following lemma on erasing real singularities of holomorphic functions in two complex variables.

Lemma 7.32 (Exercise from Task 1). *Let $D \subset \mathbb{C}^2$ be a domain intersecting $\mathbb{R}^2$. Each holomorphic function $f : D \setminus \mathbb{R}^2 \to \mathbb{C}$ extends holomorphically to all of D.*

Proof It suffices to treat the case, when $D = \Delta_{\delta,\delta} = \Delta(0, \delta)$, and prove that each holomorphic function $f : D \setminus \mathbb{R}^2$ extends holomorphically to the origin. To do this, consider the family of parabolas

$$S_t = \{w = i(z^2 + t)\} \cap \{|z| \leq \frac{\delta}{4}\},\ 0 \leq t \leq \frac{\delta}{4},\ S := S_0.$$

The sets S_t are one-dimensional compact surfaces.

Claim. *One has $S_t \cap \mathbb{R}^2 = \emptyset$ for $t > 0$; $S_0 \cap \mathbb{R}^2 = \{0\}$.*

Proof Let $(z, w) \in S_t \cap \mathbb{R}^2$. Then $z^2 + t \geq 0$, hence $w \in \mathbb{R} \cap i\mathbb{R} = \{0\}$, $w = 0 = z^2 + t$. The latter equality holds only for $z = t = 0$. This proves the claim. □

The surfaces S_t with $t > 0$ are contained in $D \setminus \mathbb{R}^2$ and converge to the surface $S = S_0$ passing through $0 \in \partial(D \setminus \mathbb{R}^2)$ with boundaries, and $\partial S \subset D \setminus \mathbb{R}^2$. Therefore, each holomorphic function on $D \setminus \mathbb{R}^2$ extends holomorphically to a neighborhood of the surface S, and hence, to the origin (Continuity Principle). This proves the lemma. □

Remark 7.33 One can prove the lemma by extending the functions to S as Cauchy integrals along the surfaces S_t, without using the Continuity Principle. That is consider the new coordinates $(z, \widetilde{w})$, $\widetilde{w} = w - iz^2$, in which the parabolas S_t are discs $\widetilde{S}_t = \{\widetilde{w} = it\}$. Then the Cauchy formula for a function f written via integrating along the boundaries ∂S_t depends holomorphically on $\widetilde{w}$ and defines a holomorphic extension of the function f to $\widetilde{S}_0$.

Exercise. Prove higher-dimensional analogue of Lemma 7.32.

Definition 7.34 Let $D \subset \mathbb{C}^n$ be a domain, $\zeta \in \partial D$. We say that D is *Levy-* (or L-) convex at ζ, if for every compact holomorphic surface $S \subset \mathbb{C}^n$ through ζ with $\partial S \subset D$ for every sequence S_k of compact holomorphic surfaces converging to S with $\partial S_k \to \partial S$ one has $S_k \setminus D \neq \emptyset$, whenever k is large enough. A domain D is called *L-convex,* if it is L-convex at each $\zeta \in \partial D$.

Proposition 7.35 *Let a domain $D \subset \mathbb{C}^n$ be not holomorphically extendable at a point $\zeta \in \partial D$. Then it is L-convex at ζ.*

Proof Suppose the contrary: D is not L-convex at a $\zeta \in \partial D$. Then there exist a compact holomorphic surface $S \subset \mathbb{C}^n$, $\zeta \in S$, $\partial S \subset D$ and a sequence $S_k \to S$ of compact holomorphic surfaces $S_k \subset D$ converging to S with boundaries. Then each holomorphic function on D extends holomorphically to a neighborhood of the limit surface S, and hence, to ζ (Continuity Principle). Thus, D is holomorphically extendable to ζ. The contradiction thus obtained proves the proposition. $\square$

The next theorem provides a global converse statement.

Theorem 7.36 (Oka). *A domain $D \subset \mathbb{C}^n$ is holomorphically non-extendable (at all the points of its boundary), if and only if it is L-convex.*

Theorem 7.37 (Sufficient condition for L-convexity). *Let $D \subset \mathbb{C}^n$ be a domain, $\zeta \in \partial D$. Let there exist a neighborhood $U = U(\zeta) \subset \mathbb{C}^n$ and a function f holomorphic on U such that*

$$f(\zeta) = 0, \ \ f|_{D \cap U} \not\equiv 0.$$

Then D is not holomorphically extendable (and hence, it is L-convex) at ζ.

Proof The function $f^{-1} = \frac{1}{f}$ is holomorphic on $U \cap D$, $f^{-1}(\zeta) = \infty$. This implies that the function $f^{-1}(w)$ does not extend holomorphically to ζ, by definition. Therefore, D is not holomorphically extendable to ζ, and hence, it is L-convex there. The theorem is proved. $\square$

7.4 Levi form. Necessary and sufficient Levi convexity conditions for domains with C^2-smooth boundary

Here we consider a domain $D \subset \mathbb{C}^n$ and a point $\zeta \in \partial D$ where the boundary is C^2-smooth. That is, there exist a neighborhood $U = U(\zeta) \subset \mathbb{C}^n$ and a C^2-function $\phi : U \to \mathbb{R}$ such that

$$D \cap U = \{\phi < 0\}, \ d\phi(\zeta) \neq 0. \tag{7.6}$$

We give necessary and sufficient conditions for L-convexity of the domain D at ζ in terms of the Hessian of the function ϕ: positive (non-negative) definiteness of an appropriate Hermitian form associated to ϕ and called the Levi form. To define it, let us recall that the differential of every complex-valued function g on a domain $U \subset \mathbb{C}^n$ is a sum of its $\mathbb{C}$-linear part and its $\mathbb{C}$-antilinear part:

$$dg = \partial g + \bar{\partial} g; \ \partial g(z) : T_z\mathbb{C}^n \to \mathbb{C} \text{ is } \mathbb{C}- \text{ linear}; \ \bar{\partial} g(z) : T_z\mathbb{C}^n \to \mathbb{C} \text{ is } \mathbb{C}- \text{ antilinear},$$

$$\partial g = \sum_{j=1}^{n} \frac{\partial g}{\partial z_j} dz_j, \ \bar{\partial} g = \sum_{j=1}^{n} \frac{\bar{\partial} g}{\partial \bar{z}_j} \overline{dz_j},$$

$$(\bar{\partial} g(z))(v) = \overline{(\partial g(z))(v)} \text{ for every } v \in T_z\mathbb{C}^n, \text{ whenever } g \text{ is real-valued.} \tag{7.7}$$

The latter statement follows from the general fact that the sum of a $\mathbb{C}$-linear and a $\mathbb{C}$-antilinear functionals is real-valued, if and only if they are complex-conjugated.

Given a C^2-function $\phi : U \to \mathbb{R}$ and a $\zeta \in U$, we define a Hermitian form $\widetilde{L}(v_1, \bar{v}_2)$ on $T_\zeta\mathbb{C}^n$ as follows. For given $v_1, v_2 \in T_\zeta\mathbb{C}^n$ let us take two arbitrary germs at ζ of holomorphic vector fields $u_1(z)$, $u_2(z)$ such that $u_j(\zeta) = v_j$. Set

$$g(z) = (\partial\phi(z))(u_1(z)), \ \psi(z) = (\bar{\partial} g(z))(u_2(z)), \ \widetilde{L}(v_1, v_2) = \psi(\zeta). \tag{7.8}$$

Proposition 7.38 *The value $\widetilde{L}(v_1, \bar{v}_2)$ is well-defined: it depends only on $v_1, v_2 \in T_\zeta\mathbb{C}^n$ and does not depend on the choice of vector fields u_j. It is given by an Hermitian form $\widetilde{L}$ on $T_\zeta\mathbb{C}^n$. In local holomorphic coordinates $(z_1, \dots, z_n)$ centered at ζ the latter Hermitian form is*

$$\widetilde{L} = \sum_{j,s=1}^{n} \frac{\partial^2\phi}{\partial \bar{z}_s \partial z_j}(\zeta) dz_j \overline{dz_s} : \tag{7.9}$$

$$\widetilde{L}(v_1, \bar{v}_2) = \sum_{j,s=1}^{n} \frac{\partial^2 \phi}{\partial \bar{z}_s \partial z_j}(\zeta) v_{1,j} \bar{v}_{2,s}, \tag{7.10}$$

where $v_j = (v_{j,1} \dots, v_{j,n})$.

Proof It suffices to prove the coordinate presentation (7.9): the well-definedness then follows immediately. One has

$$g(z) = \sum_{j=1}^{n} \frac{\partial \phi}{\partial z_j}(z) u_j(z).$$

Taking $\bar{\partial}$-differential of the latter right-hand side results in differentiating only the partial derivatives of the function ϕ: $\bar{\partial} u_j = 0$, since u_j are holomorphic. This implies that $\psi(z) = (\bar{\partial} g(z))(u_2(z))$ equals the value of the Hermitian form (7.9) on the pair of vector fields $(u_1(z), u_2(z))$. Taking the value $\psi(\zeta) = \widetilde{L}(v_1, v_2)$ yields (7.10). The proposition is proved. □

Let D, U, ζ, ϕ be the same, as in (7.6). Set

$$H = T_\zeta \partial D = \mathrm{Ker}(d\phi(\zeta)).$$

Claim. *Consider the maximal complex subspace* $H_{\mathbb{C}} \subset H$, *see Proposition 6.8. One has*

$$H_{\mathbb{C}} = K = \mathrm{Ker}(\partial \phi(\zeta)) = \mathrm{Ker}(\sum_{j=1}^{n} \frac{\partial \phi(\zeta)}{\partial z_j} dz_j).$$

Proof The latter kernel K is a complex vector subspace of complex codimension one in $T_\zeta \mathbb{C}^n$, being the kernel of a $\mathbb{C}$-linear functional. The differential $d\phi(\zeta)$ vanishes on K, being the sum of the functional $\partial \phi(\zeta)$ (annulating K) and its complex conjugate, see (7.7). Therefore, $H_{\mathbb{C}} \subset H$. This together with the latter codimension statement and Proposition 6.8 implies that $H_{\mathbb{C}}$ is the maximal complex subspace in H. The claim is proved. □

Definition 7.39 The Hermitian form $\widetilde{L}$ on $T_\zeta \mathbb{C}^n$ from Proposition 7.38, see (7.9) is called the *extended Levi form.* Its restriction

$$L = \widetilde{L}|_{H_{\mathbb{C}}}$$

is called the *Levi form.*

In what follows we will use the invariance of the (extended) Levi form under holomorphic mappings.

Proposition 7.40 *The extended Levi form associated to a function ϕ is invariant under holomorphic mappings. That is, let $W \subset \mathbb{C}^r$, $h : W \to D \subset \mathbb{C}^n$ be a holomorphic mapping, $\phi : D \to \mathbb{R}$ be a C^2-function. Let $\widetilde{L}_\phi$ and $\widetilde{L}_{\phi\circ h}$ be respectively the extended Levi forms associated to the functions ϕ and $\phi \circ h$. Then for every $z \in W$ and vectors $v_1, v_2 \in T_z\mathbb{C}^n$ one has*

$$\widetilde{L}_{\phi\circ h}(v_1, v_2) = \widetilde{L}_\phi((dh(z))(v_1), (dh(z))(v_2)). \tag{7.11}$$

Proof Consider the invariant definition (7.8) of the extended Levi form $\widetilde{L}_{\phi\circ h}$ at $\zeta \in W$ with u_j being holomorphic vector fields on a neighborhood of ζ, $u_j(\zeta) = v_j$. One has

$$g(z) = (\partial\phi(h(z)))(dh(z)u_1(z)), \ \psi(z) = (\bar{\partial}g(z))(u_2(z)).$$

The function $g(z)$ is a linear combination of partial derivatives of the function ϕ with coefficients being holomorphic functions: the components of the holomorphic vector function $(dh(z))u_1(z)$. Taking its $\bar{\partial}$-derivative along the field $u_2(z)$ results in differentiating the derivatives of the function ϕ only and subsequent multiplying them by the components of the vector function $\overline{(dh(z))u_2(z)}$, by holomorphicity. This implies (7.11) and proves the proposition. □

Theorem 7.41 (Levi–Krzoska). *Let D, U, ζ, ϕ be the same, as in (7.6). Let $H_{\mathbb{C}} \subset H = T_\zeta\mathbb{C}^n$ be the maximal complex subspace. Let L be the above-defined Levi form on $H_{\mathbb{C}}$.*

1) Let L be positive definite. Then D is holomorphically non-extendable at ζ.

2) Let D be L-convex at ζ. Then L is non-negatively definite.

The proof of the theorem will be based on the following proposition.

Proposition 7.42 *Let ϕ be a germ of real-valued C^2-function on a neighborhood of the origin in $\mathbb{C}^n$. Let $d\phi(0) \neq 0$. Let $\widetilde{L}$ denote the corresponding extended Levi form on $T_0\mathbb{C}^n$. Then in appropriate local coordinates $z = (z_1, \dots, z_n)$ centered at 0 the function ϕ takes the form*

$$\phi(z) = \operatorname{Re} z_1 + \widetilde{L}(z, \bar{z}) + o(|z|^2), \ \text{as } z \to 0. \tag{7.12}$$

Here we take the value of the extended Levi form on the Euler vector field $z = (z_1, \dots, z_n)$.

Proof Set $H = \mathrm{Ker}(d\phi(0))$. Let $H_{\mathbb{C}} \subset H$ denote the maximal complex subspace. First take preliminary local coordinates z so that $H_{\mathbb{C}}$ coincides with the $(z_2, \dots, z_n)$-subspace: $dz_1|_{H_{\mathbb{C}}} \equiv 0$. The differential $d\phi(0)$ being a non-zero $\mathbb{R}$-valued $\mathbb{R}$-linear functional vanishing on $H_{\mathbb{C}}$, it takes the form $adz_1 + \overline{adz_1}$. We can and will consider that $d\phi(0) = \mathrm{Re}\, dz_1$: one can achieve this by changing z_1 to $2az_1$. Then ϕ takes the form

$$\phi(z) = \mathrm{Re}\, z_1 + \widetilde{L}(z, \bar{z}) + Q(z, z) + \overline{Q(z, z)} + o(|z|^2).$$

Here Q is a $\mathbb{C}$-bilinear quadratic form, which we evaluate on the Euler field $z = (z_1, \dots, z_n)$:

$$Q(z, z) = \sum_{j,s=1}^{n} q_{js} z_j z_s.$$

We kill the Q-terms by the local coordinate transformation

$$z_1 = \widetilde{z}_1 - 2Q(z, z), \ z \mapsto \widetilde{z} = (\widetilde{z}_1, z_2, \dots, z_n).$$

In the new coordinates one has

$$\phi(z) = \mathrm{Re}\, \widetilde{z}_1 - 2\,\mathrm{Re}(Q(z, z)) + Q(z, z) + \overline{Q(z, z)} + \widetilde{L}(z, \bar{z}) + o(|z|^2).$$

The Q-terms obviously cancel out, and ϕ takes the form (7.12). The proposition is proved. $\square$

Proof of Theorem 7.41. Let us prove Statement 1). Let $L > 0$. Consider the local coordinates z centered at ζ, $z(\zeta) = 0$ satisfying (7.12).

Claim. *There exists a neighborhood $U = U(\zeta) \subset \mathbb{C}^n$ such that $z_1 \neq 0$ on $D \cap U$.*

Proof Set $w = (z_2, \dots, z_n)$. We have to show that the intersection of the domain D with the coordinate w-subspace $H_{\mathbb{C}}$ does not accumulate to $0 = \zeta$. Or equivalently, $\phi|_{H_{\mathbb{C}}} \geq 0$ on a neighborhood of the origin in $H_{\mathbb{C}}$. One has

$$\phi|_{H_{\mathbb{C}}} = L(w, \bar{w}) + o(|w|^2), \tag{7.13}$$

by (7.12). This together with positive definiteness of the Levi form L on $H_{\mathbb{C}}$ implies non-negativity of the latter right-hand side on a neighborhood of the origin in $H_{\mathbb{C}}$. The claim is proved. $\square$

The function z_1 vanishes at $\zeta \in \partial D$ and does not vanish on $D \cap U$. This together with Theorem 7.37 implies holomorphic non-extendability of the domain D at ζ. Statement 1) is proved.

Let us now prove Statement 2). Suppose the contrary: $L(v, \bar{v}) = -c < 0$ for some $v \in H_{\mathbb{C}}$. Let us show that D is not L-convex. Without loss of generality we consider that $v = (0, 1, 0, \dots, 0)$ (applying a linear change of coordinates $w = (z_2, \dots, z_n)$, which does not change (7.12)). For a small $\delta > 0$ set

$$S = \{|z_2| \leq \delta,\ z_1 = z_3 = \dots, z_n = 0\}.$$

We show that $\partial S \subset D$ and the compact holomorphic curve S is the limit of a family of compact holomorphic curves S_k on which $\phi < 0$, hence $S_k \subset D$. This will imply that D is not L-convex.

For every δ small enough one has $\phi|_S \leq 0$ and $\phi|_{\partial S} < 0$, by (7.13), as in the above proof of Statement 1). Hence, $\partial S \subset D$. For every $k \in \mathbb{N}$ set

$$S_k = S - (\frac{1}{k}, 0, \dots, 0): \text{ the curve } S \text{ shifted by the vector } (-\frac{1}{k}, 0, \dots, 0).$$

One has

$$\phi|_{S_k} = -\frac{1}{k} - c|z_2|^2 + o(\frac{|z_2|}{k}) + o(|z_2|^2) + o(\frac{1}{k^2}),\ c > 0,$$

by definition and (7.12). The sum of two first terms is negative and its module dominates the remaining terms. Therefore, $\phi|_{S_k} < 0$, hence $S_k \subset D$. This together with the previous discussion proves Statement 2) and the Theorem. □

7.5 Subharmonic functions and L-convexity

Levi–Krzoska Theorem gives a sufficient condition for L-convexity of a domain with C^2-smooth boundary: strict positivity of the Levi form. Here we show that a domain is automatically L-convex (and hence, a domain of holomorphy), if it is a sublevel set of a function from a specific class: the plurisubharmonic functions. The corresponding Levi forms are nonnegative definite but not necessarily strictly positive definite. The plurisubharmonic functions are natural generalizations of the subharmonic functions in one complex variable. They have important applications. For example, the proof of one of the most fundamental theorems of geometry, the Poincaré–Köbe Uniformization Theorem, is based on use of subharmonic functions.

Definition 7.43 A C^2-function $\phi : V \to \mathbb{R}$ on a domain $V \subset \mathbb{C}$ is *harmonic (subharmonic),* if for every $z_0 \in V$ and every $r > 0$ such that $D_r(z_0) \subset V$ one has

$$\phi(z_0) = \frac{1}{2\pi} \int_0^{2\pi} \phi(z_0 + re^{i\theta}) d\theta, \tag{7.14}$$

respectively,

$$\phi(z_0) \leq \frac{1}{2\pi} \int_0^{2\pi} \phi(z_0 + re^{i\theta}) d\theta. \tag{7.15}$$

Recall that the Laplacian of a function in one complex variable is expressed in terms of ∂ and $\overline{\partial}$ operators as follows:

$$\Delta = 4 \frac{\partial^2}{\partial z \partial \bar{z}}.$$

Proposition 7.44 *A C^2-function of one complex variable is harmonic, if and only if it satisfies the Laplace equation $\Delta\phi = 0$, or equivalently,*

$$\frac{\partial^2}{\partial z \partial \bar{z}} \phi = 0.$$

A C^2-function of one complex variable is subharmonic, if and only if its Laplacian is nonnegative:

$$\frac{\partial^2}{\partial z \partial \bar{z}} \phi \geq 0. \tag{7.16}$$

A C^2-function on a domain $V \subset \mathbb{C}$ is (sub)harmonic, if and only if for every point $z_0 \in V$ there exists an $r_0 > 0$ such that equality (7.14) (inequality (7.15)) holds for every $0 < r < r_0$.

Proof The first statement of the proposition is a classical theorem of analysis. Let us prove the second one, on the subharmonic functions.

Step 1): subharmonicity implies non-negativity of the Laplacian. Fix an arbitrary $z_0 \in V$, and let us prove that $\frac{\partial^2}{\partial z \partial \bar{z}} \phi(z_0) \geq 0$. Let us choose the affine coordinate z centered at z_0 and write Taylor expansion of the function ϕ at $z_0 = 0$:

$$\phi(z) = \phi(0) + az + \overline{az} + cz^2 + \overline{cz^2} + dz\bar{z} + o(|z|^2), \ a, c \in \mathbb{C}, \ d = \frac{\partial^2}{\partial z \partial \bar{z}} \phi(0) \in \mathbb{R}.$$

It suffices to show that $d \geq 0$. The non-negative difference of the right- and left-hand sides in inequality (7.15) is equal to the integral

$$\frac{1}{2\pi} \int_0^{2\pi} (\psi(z) + dz\bar{z} + o(|z|^2)) d\theta, \ z = e^{i\theta}, \ \psi(z) = 2 \operatorname{Re}(az + cz^2).$$

The integral of the function $\psi(z)$ vanishes, since ψ is a linear combination of the exponents $e^{\pm i\theta}$, $e^{\pm 2i\theta}$. The integral of the function $dz\bar{z}$ equals dr^2, and it dominates the integral of the third one. Therefore, if $d < 0$, then the total integral is negative, – a contradiction. Hence, $d \geq 0$.

Step 2): non-negativity of the Laplacian implies subharmonicity. Let $\frac{\partial^2}{\partial z\partial\bar{z}}\phi \geq 0$. Let us prove inequality (7.15) at a given point z_0. We choose affine coordinate z centered at z_0, thus we consider that $z_0 = 0$. Set

$$g(r) = \frac{1}{2\pi}\int_0^{2\pi} \phi(re^{i\theta})d\theta, \quad D_r(z_0) \subset V$$

Claim. *The function $g(r)$ is non-decreasing.*
Proof One has

$$\frac{\partial g}{\partial r} = \frac{1}{2\pi}\int_0^{2\pi}(\frac{\partial \phi}{\partial z}e^{i\theta} + \frac{\partial \phi}{\partial \bar{z}}e^{-i\theta})d\theta = \frac{1}{2\pi i}\oint_{|z|=r}(\frac{\partial \phi}{\partial z}dz - \frac{\partial \phi}{\partial \bar{z}}\overline{dz})$$

$$= -\frac{1}{\pi i}\int_{|z|<r}\frac{\partial^2}{\partial z\partial\bar{z}}\phi dz \wedge \overline{dz} = \frac{2}{\pi}\int_{|z|<r}\frac{\partial^2}{\partial z\partial\bar{z}}\phi dx \wedge dy \geq 0.$$

This proves the claim. □

One has $g(0) = \phi(0)$, hence $g(r) \geq \phi(0)$. This proves inequality (7.15) and the second step.

Note that in the proof of Step 1) we have used only the existence of an $r_0 = r_0(z_0) > 0$ such that inequality (7.15) holds for all $0 < r < r_0$. Therefore, under the latter condition the Laplacian of the function ϕ is everywhere nonnegative. This together with Step 2) implies the third statement of the proposition and finishes its proof. □

Remark 7.45 The general definition of subharmonic function does not require even continuity: only upper semicontinuity and inequality (7.15) are required. They are defined as functions with values in $\hat{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$. For example, the function $\ln|z|$ is harmonic on $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ and subharmonic on $\mathbb{C}$: the mean inequality (7.15) holds at the origin, where the function equals minus infinity. This is a continuous $\hat{\mathbb{R}}$-valued function. The series

$$\sum_{k=1}^{+\infty}\frac{1}{k^3}\ln|z - \frac{1}{k}|$$

defines a subharmonic function on $\mathbb{C}$ that is discontinuous at the origin.

Remark 7.46 The motivation of the term "subharmonic" is the following. Consider the Dirichlet problem to find a harmonic function f on a domain $V \subset \mathbb{C}$ that is continuous on its closure and satisfies the boundary condition $f|_V = \psi$, where $\psi : \partial V \to \mathbb{R}$ is a given continuous function. Let now ϕ

be a subharmonic function on V that satisfies the same boundary condition $\phi|_V = \psi$. Then $\phi \leq f$ on V. Vice versa, take an arbitrary continuous function ϕ on a domain $W \subset \mathbb{C}$. Take an arbitrary disk $D = D_r(z_0) \subset W$ and the harmonic extension $f : D \to \mathbb{R}$ of the restriction $\phi|_{\partial D}$. Let for every D as above one have $\phi \leq f$ in D. Then ϕ is subharmonic.

Theorem 7.47 (Maximum Principle for subharmonic functions). *Let $V \subset \mathbb{C}$ be a bounded domain, $\phi : V \to \mathbb{R}$ be a subharmonic function continuous on $\overline{V}$. Then*

$$\max_{\overline{V}} \phi = \max_{\partial V} \phi.$$

If ϕ achieves its maximum at an interior point $z_0 \in V$, then it is constant.

The theorem follows immediately from inequality (7.15).

Definition 7.48 Let $D \subset \mathbb{C}^n$ be a domain. A C^2 function $\phi : D \to \mathbb{R}$ is *pluri(sub)harmonic,* if for every complex line $\Lambda \subset \mathbb{C}^n$ the restriction to $\Lambda \cap D$ of the function ϕ is (sub)harmonic.

Proposition 7.49 *A C^2-function $\phi : D \to \mathbb{R}$ is pluriharmonic, if and only if the corresponding extended Levi form $\widetilde{L}$ vanishes identically. A C^2-function $\phi : D \to \mathbb{R}$ is plurisubharmonic, if and only if its extended Levi form $\widetilde{L}$ is non-negative definite at each point in D.*

Proof Let $\Lambda \subset \mathbb{C}^n$ be an arbitrary complex line. Consider a system of affine coordinates $(z_1, \ldots, z_n)$ such that L is the z_1-axis. Let $\zeta \in \Lambda$, $v_\zeta = \frac{\partial}{\partial z_1} \in T_\zeta \Lambda$ denote the unit vector directing the z_1-axis. One has

$$\frac{\partial^2 \phi}{\partial z_1 \partial \bar{z}_1}(\zeta) = \widetilde{L}(v_\zeta, \bar{v}_\zeta),$$

by definition. Therefore, the latter derivative is zero (non-negative) for all Λ and $\zeta \in \Lambda \cap D$, if and only if the extended Levi form vanishes identically (respectively, non-negative definite) at each point in D. The proposition is proved. $\square$

Corollary 7.50 *The notion of pluri(sub)harmonicity is invariant under holomorphic mappings. Namely, the composition $\phi \circ h$ of a pluri(sub)harmonic function ϕ with a holomorphic mapping h is pluri(sub)harmonic.*

The corollary follows from Propositions 7.49 and 7.40.

Corollary 7.51 *The restriction of a pluri(sub)harmonic function to a compact holomorphic surface is pluri(sub)harmonic.*

Remark 7.52 In fact, the notion of plurisubharmonic function is more general than in the above definition: it includes discontinuous functions. The general definition requires only upper semicontinuity on the definition domain and subharmonicity of restrictions to complex lines.

Theorem 7.53 (Maximum Principle for plurisubharmonic functions). *Let $\phi : D \to \mathbb{R}$ be a plurisubharmonic function, and let $S \subset D$ be a compact holomorphic surface. Then*

$$\max_S \phi = \max_{\partial S} \phi.$$

Or equivalently, if the function $\phi|_S$ achieves its maximum in the interior of the surface S, then it is constant.

Proof Let $h : \overline{W} \to S$ be a holomorphic parametrization by a domain $W \subset \mathbb{C}^r$ with compact closure. The function $g = \phi \circ h$ is plurisubharmonic. Suppose the contrary: it achieves a maximum at a point $\zeta \in Int(S)$. Then its restriction to each line through ζ is a subharmonic function achieving a local maximum at ζ and hence, is equal to the same constant $g(\zeta)$ for all lines. Finally, the function g is constant on a neighborhood of the point ζ. The above argument shows that the level set $\{g = \zeta\}$ is open, and it is closed by continuity. This together with connectivity implies that the latter level set coincides with all of W, hence $g \equiv const$. The theorem is proved. □

Theorem 7.54 *Let $D \subset \mathbb{C}^n$ be a domain, $\zeta \in \partial D$. Let there exist a neighborhood $U = U(\zeta) \subset \mathbb{C}^n$ and a plurisubharmonic function $\phi : U \to \mathbb{R}$ such that $D \cap U = \{\phi < 0\}$. Then D is L-convex at ζ.*

Proof Suppose the contrary: D is not L-convex at ζ. Then there exists a compact holomorphic surface $S \subset U$ through ζ such that $\partial S \subset D \cap U$. Thus, $\phi|_S$ is a plurisubharmonic function such that $\phi|_{\partial S} < 0$ and $\phi(\zeta) = 0$, – a contradiction to the Maximum Principle. The theorem is proved. □

Corollary 7.55 *Let $\phi : V \to \mathbb{R}$ be a plurisubharmonic function. Let $D = \{\phi < 0\} \Subset V$. Then D is L-convex, and hence, a domain of holomorphy.*

Proof The domain D is L-convex by Theorem 7.54. This together with Theorems 7.25 and 7.36 implies that D is a domain of holomorphy. □

8 Stein manifolds, Dolbeault cohomology and sheaves

8.1 Stein manifolds: definition and main properties

Definition 8.1 A complex manifold M is said to be *holomorphically convex*, if the holomorphically convex hull ($H(M)$-hull) of each its compact subset is a compact subset. We say that the holomorphic functions on M *separate points,* if for every two distinct points $x \neq y$ in M there exists a holomorphic function $f : M \to \mathbb{C}$ such that $f(x) \neq f(y)$.

Definition 8.2 A complex manifold M, set $n = dimM$, is said to be a *Stein manifold,* if it satisfies the following conditions:

1) M is holomorphically convex;
2) the holomorphic functions on M separate points;
3) for every $z \in M$ there exist n holomorphic functions $f_1, \dots, f_n$ on M whose differentials at z are linearly independent: that is, the holomorphic vector function $(f_1, \dots, f_n) : M \to \mathbb{C}^n$ is a local biholomorphism at z.

Example 8.3 Every domain of holomorphy in $\mathbb{C}^n$ is obviously Stein. Let now $M \subset \mathbb{C}^N$ be a holomorphic submanifold. Then it is Stein. Indeed, let H denote the collection of projections $M \to \mathbb{C}$ to the coordinate axes, which are holomorphic functions. The functions from class H obviously separate points. Condition 3) holds even for functions from class H. Finally, the H-convex hull of each compact subset in M is a bounded closed subset in $\mathbb{C}^N$ contained in M. Hence, it is compact. The next big theorem states the converse.

Theorem 8.4 (Embedding Theorem). *Each Stein manifold can be embedded as a submanifold in $\mathbb{C}^N$ for appropriate N. This is true for $N = 2n + 2$, where n is the dimension of the manifold.*

Corollary 8.5 *Every domain of holomorphy in $\mathbb{C}^n$ can be embedded as a submanifold in $\mathbb{C}^{2n+2}$. In particular, every geometrically convex domain in $\mathbb{C}^n$ can be embedded as a submanifold in $\mathbb{C}^{2n+2}$.*

Example 8.6 The unit disk $D_1 \subset \mathbb{C}$ is Stein. It admits an embedding as a submanifold in $\mathbb{C}^N$. The proof of this statement is non-trivial. An exercise from Task 4 asks to prove the existence of its embedding to $\mathbb{C}^2$ by using polynomial automorphisms with Fatou–Bieberbach domains.

We will not prove Theorem 8.4 in full generality. We will sketch its proof and prove some parts of it. The proof uses Dolbeault cohomology and sheaves. The corresponding background material will be presented later on.

In what follows we will provide another important example of Stein manifolds: envelopes of holomorphy. They form a special class of Riemann domains: complex manifolds that admit locally biholomorphic projection to $\mathbb{C}^n$. They are the domains of maximal holomorphic extension of all holomorphic functions on a given Riemann domain.

8.2 Riemann domains and their holomorphic extensions

Definition 8.7 A n-dimensional *Riemann domain* is a pair (X, π), where X is a connected Hausdorff topological space with countable base and $\pi : X \to \mathbb{C}^n$ is a continuous mapping that is a local homeomorphism: each point $x \in X$ has a neighborhood $U = U(x) \subset X$ such that the restriction $\pi|_U$ is a local homeomorphism of the domain U onto a neighborhood of the image $\pi(x)$. The maximal number of preimages of a point in $\mathbb{C}^n$ will be called the *number of sheets*.

Remark 8.8 A Riemann domain has a natural structure of n-dimensional complex manifold lifted from the base. That is, the holomorphic atlas is formed by the above neighborhoods $U(x)$, which are identified by π with domains in $\mathbb{C}^n$. The projection π is holomorphic.

Example 8.9 A domain in $\mathbb{C}^n$ is a trivial example of Riemann domain with identical projection and one sheet. Another example is the double covering (two-sheeted Riemann domain)

$$\pi : \mathbb{C}^2 \setminus \{w = 0\} \to \mathbb{C}^2 \setminus \{w = 0\} : \ (z, w) \mapsto (z, w^2). \tag{8.1}$$

The latter example can be generalized as follows. Let us replace the above deleted z-axis by an arbitrary real two-dimensional subspace $L \subset \mathbb{C}^2$, e.g., $L = \mathbb{R}^2$. The connected double covering over $\mathbb{C}^2 \setminus L$ is a Riemann domain homeomorphic to $\mathbb{C}^2 \setminus \{w = 0\}$. But as the next proposition shows, its structure of complex manifold could be different.

Definition 8.10 Let (X, π_X), (Y, π_Y) be Riemann domains over $\mathbb{C}^n$. A *Riemann domain mapping* $\phi : (X, \pi_X) \to (Y, \pi_Y)$ is a continuous mapping $\phi : X \to Y$ that forms a commutative diagram with the projections

$$\pi_X = \pi_Y \circ \phi :$$

ϕ is a holomorphic mapping of complex manifolds, by definition.

In what follows for a Riemann domain X by $H(X)$ we will denote the space of holomorphic functions on the complex manifold X. Let $S \subset H(X)$ be a subset.

Definition 8.11 An S-*extension* of a Riemann domain (X, π_X) is a pair of a Riemann domain (Y, π_Y) and a Riemann domain mapping of $\phi : (X, \pi_X) \to (Y, \pi_Y)$ such that for every $f \in S$ there exists a function $F \in H(Y)$ such that $f = F \circ \phi$. In the case, when $S = H(X)$, we call the pair $((Y, \pi_Y), \phi)$ a *holomorphic extension.*

Example 8.12 Let $\Omega \subset \Delta$ be a Hartogs figure in a polydisk Δ. Then Δ is a holomorphic extension of the domain Ω, by Hartogs' Theorem.

In what follows we discuss two examples: a double-sheeted Riemann domain with a one-sheeted holomorphic extension; a one-sheeted domain with two-sheeted holomorphic extension.

Proposition 8.13 *Consider the Riemann domain (X, π_X) that is a connected double covering over $\mathbb{C}^2 \setminus \mathbb{R}^2$. Then for every function $f \in H(X)$ there exists a holomorphic function $F : \mathbb{C}^2 \to \mathbb{C}$ such that $f = F \circ \pi$. In other words, the trivial one-sheeted Riemann domain $(\mathbb{C}^2, Id)$ is a holomorphic extension of the two-sheeted domain (X, π_X) via the mapping $\phi = \pi_X : X \to \mathbb{C}^2$.*

Proof Let (z, w) denote the standard coordinates on $\mathbb{C}^2$. Let us introduce the new coordinates:

$$(z, \widetilde{w}), \ \widetilde{w} = w - iz^2.$$

Set

$$V = \{\operatorname{Im} \widetilde{w} > -|z|^2\} \subset \mathbb{C}^2.$$

We show that

1) V does not intersect $\mathbb{R}^2$ and is simply connected;

2) each function holomorphic on V extends holomorphically to all of $\mathbb{C}^2$.

Statement 1) implies that there exists a continuous inverse mapping $\pi^{-1} : V \to X$. Statement 2) implies that for every $f \in H(X)$ the function $F = f \circ \pi^{-1}|_V$ extends holomorphically to $\mathbb{C}^2$. It satisfies the equality $f = F \circ \pi$ on $\pi^{-1}(V)$ and hence, everywhere by uniqueness of analytic extension. This will prove the proposition.

Proof of Statement 1). Let us first show that $V \cap \mathbb{R}^2 = \emptyset$. Suppose the contrary: there exists a point $p \in V \cap \mathbb{R}^2$. Let (z, w) denote its standard coordinates: then $z, w \in \mathbb{R}$. One has

$$\operatorname{Im} \widetilde{w} = \operatorname{Im}(w - iz^2) > -|z|^2, \tag{8.2}$$

by the definition of the domain V. The left-hand side in (8.2) equals $-z^2 = -|z^2|$, since $z, w \in \mathbb{R}$. Hence, (8.2) takes the form $-|z|^2 > -|z|^2$, – a contradiction. Hence, $V \cap \mathbb{R}^2 = \emptyset$.

The mapping $(z, \widetilde{w}) \mapsto (z, t)$, $t = \widetilde{w} + i|z|^2$ sends V diffeomorphically onto the direct product $\mathbb{C} \times \{\operatorname{Im} t > 0\}$. Therefore, V is contractible. Statement 1) is proved. □

Proof of Statement 2). For every $r > 0$ set

$$W_r = D_r \times \{\operatorname{Im} \widetilde{w} > -\frac{r^2}{4}\},$$

$$A_r = \{\frac{r}{2} < |z| < r\} \times \{\operatorname{Im} \widetilde{w} > -\frac{r^2}{4}\},$$

$$B_r = D_r \times \{\operatorname{Im} \widetilde{w} > 0\}, \ \Omega_r = A_r \cup B_r \subset W_r.$$

One has $\Omega_r \subset V$, by definition. On the other hand, Ω_r is a Hartogs-like figure in W_r: each function holomorphic on Ω_r extends holomorphically to W_r, as in Hartogs' Theorem. The domains W_r form an increasing family of domains exhausting $\mathbb{C}^2$, as $r \to +\infty$. The three latter statements together imply that every function holomorphic on V extends holomorphically to all of them, and hence, to $\mathbb{C}^2$. Statement 2) is proved. □

Statements 1) and 2) imply the proposition, as was shown above. □

Remark 8.14 The Riemann domain given by the double covering (8.1) over the complement to a complex line in $\mathbb{C}^2$ obviously does not satisfy the statement of Proposition 8.13. Namely, the coordinate w of the covering space takes different values ± 1 on two distinct preimages $(0, \pm 1)$ of the point $(0, 1)$.

Now let us construct a domain $D \subset \mathbb{C}^2$ (one-sheeted Riemann domain) that admits a two-sheeted holomorphic extension. To do this, consider the parallelogram Π with the vertices $(-4, 0)$, $(0, 1)$, $(4, 1)$, $(0, 0)$ drawn on the plane

$$(x, t) \in \mathbb{R}^2, \ x = \operatorname{Re} z, \ t = |w|,$$

see Fig. 3. Set

$$\widetilde{D} = \{(z, w) \mid (\operatorname{Re} z, |w|) \in \Pi\}.$$

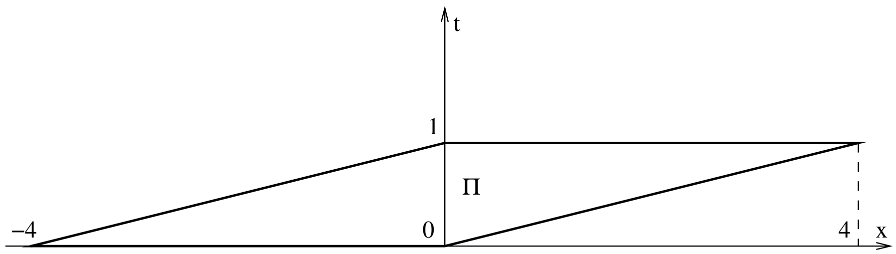

Figure 3: The parallelogram Π: $\widetilde{D} = \{(z,w) \mid (\operatorname{Re} z, |w|) \in \Pi\}$.

Consider now the projection

$$\pi : \mathbb{C}^2 \to \mathbb{C}^2,\ (z,w) \mapsto (e^{iz}, w).$$

Proposition 8.15 *The restriction $\pi|_{\widetilde{D}}$ is a bijection of the domain $\widetilde{D}$ onto its image*

$$D = \pi(\widetilde{D}) \subset \mathbb{C}^2.$$

Thus, $(\widetilde{D}, \pi)$ is a one-sheeted Riemann domain.

Proof The proposition is equivalent to the statement that the parallelogram Π is disjoint from its images under the action of the group of translations $(x,t) \mapsto (x + 2\pi m, t)$, $m \in \mathbb{Z}$. The latter statement follows from construction and the inequality $2\pi > 4$. This proves the proposition. □

Let us also consider the trapezoid Q in the (x,t)-plane with vertices $(-4,0)$, $(0,1)$, $(4,1)$, $(4,0)$, which contains Π. Consider the corresponding domain

$$W = \{(z,w) \mid (\operatorname{Re} z, |w|) \in Q\} \supset \widetilde{D},$$

see Fig. 4.

Proposition 8.16 *The projection $\pi : W \to \mathbb{C}^2$ is a two-sheeted Riemann domain. Every function holomorphic on one-sheeted Riemann domain $\widetilde{D}$ extends holomorphically to W.*

Proof The trapezoid Q intersects its translation image $Q + (2\pi, 0)$, by construction and the inequality $2\pi < 8$. This implies that the similar translation image of the domain W intersects W. This implies that the projection $\pi : W \to \mathbb{C}^2$ is non-injective. It is easy to show that the latter projection is two-sheeted, since $Q + (2\pi m, 0)$ does not intersect Q for $m \neq 0, \pm 1$, by the inequality $4 < 2\pi < 8$. This proves the first statement of the proposition.

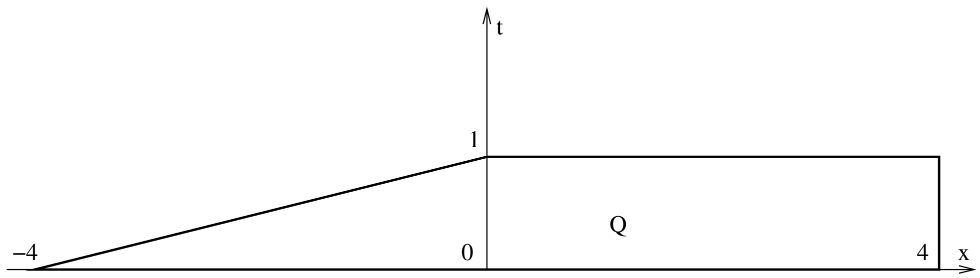

Figure 4: The trapezoid $Q \supset \Pi$: $W = \{(z, w) \mid (\operatorname{Re} z, |w|) \in Q\} \supset \widetilde{D}$.

Let us prove its second statement. Fix arbitrary

$$0 < r < 1, \ 0 < \delta < \min\{r, 1-r\}.$$

Set

$$W_{r,\delta} = 4(r+\delta-1) < \operatorname{Re} z < 4(r-\delta), \ |w| < r+\delta\},$$

$$A_{r,\delta} = \{4(r+\delta-1) < \operatorname{Re} z < 4(r-\delta), \ r-\delta < |w| < r+\delta\},$$

$$B_{r,\delta} = \{4(r+\delta-1) < \operatorname{Re} z < 0, \ |w| < r+\delta\}, \ H_{r,\delta} = A_{r,\delta} \cup B_{r,\delta}.$$

One has

$$H_{r,\delta} \subset W_{r,\delta} \cap \widetilde{D}, \ W_{r,\delta} \subset W.$$

The domain $H_{r,\delta}$ is a kind of Hartogs' figure in $W_{r,\delta}$. That is, every function holomorphic on $H_{r,\delta}$ extends holomorphically to $W_{r,\delta}$, as in Hartogs' Theorem. In particular, this is true for holomorphic functions on $\widetilde{D} \supset H_{r,\delta}$ for every r and δ as above. Taking r close to 1 and δ small enough we get that the domain $W_{r,\delta}$ covers arbitrary given compact subset in the product $U = \{0 < \operatorname{Re} z < 4\} \times \{|w| < 1\}$. This implies that each function holomorphic on $\widetilde{D}$ extends to a function holomorphic on the union $W = \widetilde{D} \cup U$. This proves the proposition. $\square$

Definition 8.17 Consider two S-extensions $\phi_j : (X, \pi_X) \to (Y_j, \pi_j)$, $j = 1, 2$. We say that $\phi_1 \leq \phi_2$, if ϕ_1 is induced from the extension ϕ_2. That is, there exists a Riemann domain mapping $\psi : (Y_1, \pi_1) \to (Y_2, \pi_2)$ such that

$$\psi \circ \phi_1 = \phi_2.$$

Remark 8.18 In the above definition one has $\pi_1 = \pi_2 \circ \psi$. The mapping ψ is an S-extension. In more detail, let $f \in S$, $F_j : Y_j \to \mathbb{C}$ be holomorphic functions such that $f = F_j \circ \phi_j$. Then

$$F_1 = F_2 \circ \psi. \tag{8.3}$$

This follows from the equality

$$f = F_1 \circ \phi_1 = F_2 \circ \phi_2 = F_2 \circ \psi \circ \phi_1,$$

which implies that (8.3) holds on $\phi_1(X) \subset Y_1$, and hence, on all of Y_1, by connectivity and uniqueness of analytic extension.

Definition 8.19 The *S-envelope of holomorphy* of a Riemann domain (X, π_X) is its maximal S-extension. In the case, when $S = H(X)$, it is called the *envelope of holomorphy.*

Remark 8.20 Let $G : (X, \pi_X) \to (Y, \pi_Y)$ be an S-envelope of holomorphy. Then for every S-extension $\phi : (X, \pi_X) \to (Z, \pi_Z)$ there exists a Riemann domain mapping $\psi : (Z, \pi_Z) \to (Y, \pi_Y)$ such that $G = \psi \circ \phi$.

Example 8.21 Let $D \subset \mathbb{C}^n$ be a domain of holomorphy: there exists a holomorphic function on it that extends analytically to no point of the boundary. Then the Riemann domain (D, Id) is its own envelope of holomorphy. In particular, this holds in the case, when $D = \mathbb{C}^n$.

Example 8.22 Let $\Omega \subset \Delta$ be a Hartogs figure in a polydisk Δ such that every holomorphic function on Ω extends holomorphically to Δ. Then $Id : (\Omega, Id) \to (\Delta, Id)$ is an envelope of holomorphy. This follows from the statement of the previous example and the fact that Δ is a domain of holomorphy, by holomorphic convexity and Oka's Theorem 7.10.

Example 8.23 Consider the Riemann domain (X, π_X) that is a connected double covering over $\mathbb{C}^2 \setminus \mathbb{R}^2$. Then $\pi_X : (X, \pi_X) \to (\mathbb{C}^2, Id)$ is its envelope of holomorphy. This follows from Proposition 8.13 and the last statement of Example 8.21.

Remark 8.24 Let (X, π_X) be a Riemann domain, $S \subset H(X)$. If its S-envelope of holomorphy exists, then it is unique up to isomorphism. Indeed, let $G_j : (X, \pi_X) \to (Y_j, \pi_j)$, $j = 1, 2$, be two S-envelopes of holomorphy. Then there exist Riemann domain mappings

$$\phi : (Y_1, \pi_1) \to (Y_2, \pi_2), \ \psi : (Y_2, \pi_2) \to (Y_1, \pi_1), \ G_2 = \phi \circ G_1, \ G_1 = \psi \circ G_2,$$

by the definition of S-envelope. The two last equalities imply that $G_1 = \psi \circ \phi \circ G_1$. Therefore, $\psi \circ \phi = Id$ on $G_1(X)$, hence, on all of Y_1. Similarly, $G_2 = \phi \circ \psi \circ G_2$ and $\phi \circ \psi = Id$. Hence, the mapping ϕ is an isomorphism of Riemann domains: the S-envelopes (Y_j, π_j).

Theorem 8.25 (Thullen). *For every Riemann domain (X, π_X) and every $S \subset H(X)$ its S-envelope of holomorphy exists.*

Proof Let (X, π_X) be a Riemann domain, $n = dimX$.

Case 1): $S = \{f\}$, $f \in H(X)$. We construct the S-envelope analogously to the construction of the Riemann surface for the maximal analytic extension of a holomorphic function of one variable. To do this, let us consider the following space of germs.

In what follows by (g, z_0) we denote the germ of a function g at z_0. Set

$$\mathcal{J} = \{(g, z_0) \mid z_0 \in \mathbb{C}^n,\ g \text{ is holomorphic on a neighborhood of } z_0\}.$$

We introduce the topology on $\mathcal{J}$ as follows. Let $(g, z_0) \in \mathcal{J}$, $U = U(z_0) \subset \mathbb{C}^n$ be a neighborhood where g is holomorphic. Set

$$W_U(g, z_0) = \{(g, z) \mid z \in U\} \subset \mathcal{J}. \tag{8.4}$$

The sets $W_U(g, z_0)$ taken for all $(g, z_0) \in \mathcal{J}$ form a base of topology. There is a natural continuous locally homeomorphic projection

$$\pi : \mathcal{J} \to \mathbb{C}^n : (g, z) \mapsto z.$$

Proposition 8.26 *The topological space $\mathcal{J}$ thus obtained is Hausdorff.*

Proof We have to show that any two distinct germs $(g_1, z_1) \neq (g_2, z_2)$ have disjoint neighborhoods. In the case, when $z_1 \neq z_2$ this statement is obvious. Let $z_1 = z_2 = z$. Then for every connected neighborhood $U = U(z) \subset \mathbb{C}^n$ where both g_1 and g_2 are holomorphic the equality $g_1 = g_2 = 0$ does not hold on an open subset in U. Indeed, otherwise, $g_1 \equiv g_2$ on U, by uniqueness of analytic extension, hence $(g_1, z) = (g_2, z)$, – a contradiction. Thus, the open sets $W_U(g_1, z)$, $W_U(g_2, z)$ are disjoint. This proves the proposition. □

Fix a point $x \in X$. Let

$$U = U(x) \subset X,\ V = V(\pi_X(x)) \subset \mathbb{C}^n$$

be the neighborhoods that are homeomorphic under the projection π_X. Let $\pi_{X,x}^{-1}$ denote its inverse sending $\pi_X(x)$ to x, thus V to U. The function

$$F_x = f \circ \pi_{X,x}^{-1}$$

is holomorphic on V. Set

$$G : X \to \mathcal{J},\ G(x) = (F_x, \pi_X(x)) \text{ for every } x \in X,$$

$$Y = \text{ the connected component of } G(X) \text{ in } \mathcal{J},\ \pi_Y = \pi|_Y.$$

The pair (Y, π_Y) is a Riemann domain, and $G : (X, \pi_X) \to (Y, \pi_Y)$ is an S-extension. Let us prove its maximality, i.e., that it is an S-envelope of holomorphy.

Let $\psi : (X, \pi_X) \to (Z, \pi_Z)$ be another S-extension. That is, there exists a holomorphic function $F : Z \to \mathbb{C}$ such that $f = F \circ \psi$. Let us construct a Riemann domain mapping $\phi : (Z, \pi_Z) \to (Y, \pi_Y)$ such that

$$G = \phi \circ \psi. \tag{8.5}$$

To do this, fix a point $x_0 \in X$, set $z_0 = \psi(x_0) \in Z$. For every $z \in Z$ let $\pi_{Z,z}^{-1} = \pi_Z^{-1}$ denote the inverse defined as above, which sends $\pi_Z(z)$ to z. Set

$$\phi(z) = (F \circ \pi_{Z,z}^{-1}, \pi_Z(z)) \in \mathcal{J}.$$

The mapping ϕ is continuous and open and $\pi_Z = \pi \circ \phi$. One has $\phi(Z) \subset Y$. This follows from the connectedness of the space Z and the equality

$$(F \circ \pi_{Z,z}^{-1}, \pi_Z(z_0)) = (F \circ \psi \circ \pi_{X,x_0}^{-1}, \pi_X(x_0)) = (f \circ \pi_{X,x_0}^{-1}, \pi_X(x_0)),$$

which implies that the latter germ and $\phi(Z)$ lie in the same connected component Y of the space $\mathcal{J}$.

The mapping $\phi : Z \to Y$ thus constructed is a mapping of Riemann domains that satisfies (8.5), by definition. Thus, $\psi \leq G$ and hence, $G : (X, \pi_X) \to (Y, \pi_Y)$ is an S-envelope of holomorphy.

Case 2): general. The proof repeats the above arguments but for the space $\mathcal{J}$ replaced by another space $\mathcal{J}_S$ of germs of S-families of holomorphic functions. Namely, we consider families of germs

$$(g_s, z_0)|_{s \in S},\ g_s \text{ are holomorphic in the same neighborhood } U = U(z_0) \in \mathbb{C}^n;$$

they are called *S-families of germs of holomorphic functions.* Two S-families (g_s, z_0) and (h_s, z_0) are *equivalent,* if there exists a neighborhood $U = U(z_0)$ where g_s are holomorphic and $g_s \equiv h_s$. The space $\mathcal{J}_S$ mentioned above is the space of S-germs with the topology defined as in the space $\mathcal{J}$, see Case 1). Theorem 8.25 is proved. $\square$

Theorem 8.27 (Oka). *The envelope of holomorphy of every Riemann domain is a holomorphically convex manifold.*

Proposition 8.28 (Task 5, Part 1, Problem 5). *The holomorphic functions on every envelope of holomorphy separate points.*

Corollary 8.29 *Every envelope of holomorphy is a Stein manifold.*

Proof Let (Y, π_Y) be an n-dimensional envelope of holomorphy. The coordinate functions $f_j = z_j \circ \pi_Y : Y \to \mathbb{C}$ are holomorphic and have linearly independent differentials at each point. This together with the above theorem and proposition implies that Y is a Stein manifold. □

8.3 One-dimensional $\bar{\partial}$-problem

The $\bar{\partial}$-problem has the following versions:

- Given a function $g(z)$ of one variable, find a function f such that

$$\frac{\bar{\partial} f}{\partial \bar{z}} = g. \tag{8.6}$$

- Given a differential m–form ω find an $(m-1)$-form α such that

$$\bar{\partial}\alpha = \omega. \tag{8.7}$$

The results on $\bar{\partial}$-problem form an important base for many famous results in complex analysis and related topics such as quasiconformal mappings, Teichmüller theory, moduli spaces of Riemann surfaces, algebraic geometry. It has many applications in the above-mentioned domains and complex dynamics.

First we treat the $\bar{\partial}$-problem for functions. Then we introduce $\bar{\partial}$-operator acting on differential forms, which defines Dolbeault complex and cohomology. We solve the corresponding $\bar{\partial}$-problem on polydisk by proving triviality of Dolbeault cohomology ($\bar{\partial}$-Poincaré Lemma). Afterwards we apply this result together with elements of sheaf theory to show that each hypersurface in a polydisk is the zero locus of a global holomorphic function.

Theorem 8.30 *For every C^∞ function $g : D_1 \to \mathbb{C}$ on the unit disk $D_1 \subset \mathbb{C}$ there exists a C^∞ function $f : D_1 \to \mathbb{C}$ satisfying (8.6).*

Addendum. *Let the function $g = g(z, w_1, w_2)$ depend on additional parameters (w_1, w_2), w_1 being a point of a polydisk $\Delta_R \subset \mathbb{C}^m$, w_2 being a point of some real manifold. Let g be C^∞-smooth in (z, w_1, w_2) and holomorphic in w_1. Then the corresponding function f can be chosen from the same class: C^∞-smooth in (z, w_1, w_2) and holomorphic in w_1.*

The addendum will be further applied to prove the above-mentioned $\bar{\partial}$-Poincaré Lemma. The proof of the theorem and the addendum will be split into two steps: construction of a function f on arbitrary smaller disk; extension to the whole disk by appropriate passing to limit along a sequence of exhausting disks; adjusting the construction to achieve regularity in parameters. In the proof we will use the following obvious remark.

Remark 8.31 If the function f solving (8.6) exists, then it is unique up to addition of a holomorphic function.

Proposition 8.32 *For every C^∞ function $g : D_1 \to \mathbb{C}$ and every $0 < r < 1$ the function*

$$f(z) = \frac{1}{2\pi i} \int_{D_r} \frac{g(\zeta)}{\zeta - z} d\zeta \wedge \overline{d\zeta}. \tag{8.8}$$

is C^∞-smooth in $z \in D_r$ and satisfies (8.6). In the case, when g depends on additional parameters as in the addendum, the function f satisfies the statements of the addendum on the disk D_r.

Proof The above integral converges and is a well-defined continuous function, being an integral of a function $O(\frac{1}{\zeta - z})$ over a real two-dimensional domain. Fix arbitrary $0 < \mu < \nu < r < 1$. Let $\rho_1 : \mathbb{C} \to \mathbb{R}$ be a C^∞-smooth function such that $\rho_1|_{D_\mu} \equiv 1$ and $\rho_1|_{\mathbb{C} \setminus D_\nu} \equiv 0$. Set

$$\rho_2 = 1 - \rho_1; \ \rho_2|_{D_\mu} \equiv 0, \ \rho_2|_{\mathbb{C} \setminus D_\nu} \equiv 1, \ g_j = \rho_j g, \ g = g_1 + g_2.$$

The integral (8.8) is the sum of the same integrals denoted f_j, $j = 1, 2$, with the function g replaced by g_j: $f = f_1 + f_2$. The integral f_2 is in fact an integral over the annulus $D_r \setminus D_\mu$, since $g_2 = 0$ on D_μ. Therefore, it is a holomorphic function in $z \in D_\mu$, being an integral of a family of holomorphic functions $\frac{1}{\zeta - z}$ in a parameter $\zeta \notin D_\mu$. Hence, $\bar{\partial} f_2 = 0$ on D_μ. Now for the proof of the proposition it suffices to show that

$$\frac{\bar{\partial} f_1}{\partial \bar{z}} = g_1. \tag{8.9}$$

The function g_1 is defined on D_r and vanishes on the annulus $D_r \setminus D_\nu$. Therefore, we can and will extend it to $\mathbb{C}$ as $g_1|_{\mathbb{C}\setminus D_r} = 0$. One has

$$f_1(z) = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{g_1(\zeta)}{\zeta - z} d\zeta \wedge \overline{d\zeta} = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{g_1(u+z)}{u} du \wedge \overline{du}, \ u = \zeta - z. \quad (8.10)$$

Taking the $\frac{\bar\partial}{\partial\bar z}$-derivative yields

$$\frac{\bar\partial f_1}{\partial \bar z}(z) = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\bar\partial g_1}{\partial \bar z}(u+z)\frac{du}{u} \wedge \overline{du}$$

$$= -\frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\bar\partial g_1}{\partial \bar z}(u+z)\overline{du} \wedge \frac{du}{u} = -\frac{1}{2\pi i}\int_{\mathbb{C}} d(g_1(u+z)\frac{du}{u})$$

The latter integral coincides with the same integral but taken over the disk D_2, since the subintegral form vanishes outside it, as does g_1. It equals the integral of the form $g_1(u+z)\frac{du}{u}$ over the boundary ∂D_2 minus the limit of its integral over the circle ∂D_δ, by Stokes formula. Taking into account that the former integral vanishes, as does $g_1(u+z)$ on ∂D_2, one has

$$\frac{\bar\partial f_1}{\partial \bar z}(z) = \lim_{\delta\to 0}(\frac{1}{2\pi i}\oint_{\partial D_\delta} g_1(u+z)\frac{du}{u}).$$

The expression under the limit is equal to the mean value of the function g_1 over the circle of radius δ centered at z; this follows by substitution $u = \delta e^{i\theta}$. Therefore, the limit equals $g_1(z) = g(z)$. This proves (8.9). The function f_1 is C^∞, since the integral in (8.10) can be differentiated: the derivatives of the subintegral expression have converging integrals, since the differentiations do not affect the denominator u. The same argument proves smoothness (holomorphicity) of the function f_1 in the additional parameters of the function g, taking into account that $g_1 = \rho_1 g$, where ρ_1 is parameter-independent. The function $f_2 = f - f_1$ is holomorphic in $z \in D_\mu$, as was shown above, and it is obviously smooth (holomorphic) in the parameters of the function g: it is defined by an integral over the annulus $D_r \setminus D_\mu$, the derivatives of the subintegral expression are well-defined and uniformly bounded, as $\zeta \in D_r \setminus D_\mu$ and z lies in a compact subset in D_μ. Finally, the function f given by the integral (8.8) has appropriate regularity in the product of the disk D_μ and the parameter spaces for every $0 < \mu < r$ and satisfies (8.6) in D_μ. The number μ can be taken arbitrarily close to r. Hence, the same statements hold with D_μ replaced by D_r. The proposition is proved. $\square$

Proof of Theorem 8.30. Set

$$r_n = 1 - \frac{1}{2^n}, \ f_n(z) = \frac{1}{2\pi i}\int_{D_{r_{n+1}}} \frac{g(\zeta)}{\zeta - z} d\zeta \wedge \overline{d\zeta}; \ \frac{\bar\partial f_n}{\partial \bar z} = g \text{ on } D_{r_{n+1}}. \quad (8.11)$$

In the case, when the functions f_n converge with all the derivatives uniformly on compact subsets in D_1 to a function f, the latter satisfies the statements of the theorem. Let us show that one can modify the functions f_n inductively, replacing them by

$$\widetilde{f}_n = f_n - P_n, \ P_n(z) \text{ being a polynomial,}$$

so that the new functions $\widetilde{f}_n$ converge. One has

$$\bar\partial \widetilde{f}_n = \bar\partial f_n = g \text{ on } D_{r_{n+1}}, \quad (8.12)$$

by construction. This together with the above argument will prove the theorem.

Induction base: $n = 1$. Set $\widetilde{f}_1 = f_1$.

Induction step. Let $\widetilde{f}_n$ be already constructed. Let us construct $\widetilde{f}_{n+1}$ so that

$$|\widetilde{f}_{n+1} - \widetilde{f}_n| < \frac{1}{2^n} \text{ on } \overline{D}_{r_n}. \quad (8.13)$$

The difference $\psi_n = f_{n+1} - \widetilde{f}_n$ is holomorphic on $D_{r_{n+1}}$, by (8.12). Therefore, its Taylor series converges to it uniformly on $\overline{D}_{r_n} \Subset D_{r_{n+1}}$. Fix a Taylor polynomial P_n such that

$$|\psi_n - P_n| < \frac{1}{2^n} \text{ on } \overline{D}_{r_n}. \quad (8.14)$$

Set $\widetilde{f}_{n+1} = f_{n+1} - P_n$. Then inequality (8.13) holds by construction. The induction step is over. The functions $\widetilde{f}_n$ are constructed.

For every compact subset $K \Subset D_1$ there exists an $N > 0$ such that the functions $\widetilde{f}_n$ with $n > N$ are well-defined on K and converge uniformly on K, as $n \to \infty$. This follows from inequality (8.13), which implies that for every $n, m > N$ the difference $\widetilde{f}_n - \widetilde{f}_m$ has module less than $\frac{1}{2^l}$, $l = \min\{m, n\} - 1$ on K. This together with the above discussion proves the theorem. $\Box$

Proof of the addendum. The functions f_n given by (8.11) have the same regularity in parameters, as g, by Proposition 8.32. Let us show that the above construction of the functions $\widetilde{f}_n$ can be done so that they have the same regularity and converge with derivatives uniformly on compact subsets in the product of the disk D_1 and the parameter space. To do this, we use the following proposition.

Proposition 8.33 *Let a function $f(w, s)$ be C^∞-smooth in the product $\Delta_R \times S$, where $\Delta_R \subset \mathbb{C}^n$ is a polydisk, $R = (R_1, \dots, R_n)$, $R_j > 0$, and S is a manifold. Let f be holomorphic in $w \in \Delta_R$. Then its Taylor series in powers of w with coefficients depending on s converges to f uniformly on compact sets together with all the derivatives.*

Proof Let us first prove uniform convergence of the Taylor series. Fix $0 < \mu < \nu < 1$ and a compact subset $K \Subset S$. Set

$$M = \max_{\overline{\Delta_{\nu R}} \times K} |f|, \ R = (R_1, \dots, R_n).$$

Let $c_k(s)$ denote the Taylor coefficient of the function f at w^k. One has

$$|c_k(s)| \leq \frac{M}{(\nu R)^k} \text{ for every } s \in K,$$

by the Cauchy Inequalities. Therefore, for every $z \in \overline{\Delta_{\mu R}}$ one has

$$|c_k(s) z^k| \leq M \beta^{|k|}, \ \beta = \frac{\nu}{\mu} < 1.$$

The right-hand sides of the above inequality form a series converging to $M(1 - \beta)^{-n}$. This implies the uniform convergence of the Taylor series on $\overline{\Delta_{\mu R}} \times K$. The convergence of derivatives is proved by the same argument with f being replaced by its derivatives. The proposition is proved. □

Let S denote the space of the variables w_2. Let Δ denote the polydisk that is the product of the z-disk D_1 and the polydisk Δ_R of the variables w_1. Fix a compact exhaustion $K_1 \Subset K_2 \Subset \dots = S$. Set

$$\delta_n = (1 - \frac{1}{2^n}) R, \ R = (R_1, \dots, R_n).$$

In the above construction of the functions $\widetilde{f_n}$ let us choose the polynomial P_n so that inequality (8.14) holds on the product $\overline{D_{r_n}} \times \overline{\Delta_{\delta_n}} \times K_n$ for the function $\psi_n - P_n$ and all its mixed derivatives in the variable z and the parameters up to order n. Then the functions $\widetilde{f_n}$ thus constructed converge with derivatives uniformly on compact sets. The limit is C^∞-smooth and is holomorphic in w_1, by Weierstrass Theorem. The addendum is proved. □

8.4 Dolbeault cohomology and $\bar{\partial}$-Poincaré Lemma

Let M be a complex manifold, $n = dim M$. For every $x \in M$ let $\Lambda_x = \Lambda_x^1$ denote the space of $\mathbb{R}$-linear complex valued functionals on T_xM, which is a $2n$-dimensional complex vector space. Each functional under question is a sum of its $\mathbb{C}$-linear part and $\mathbb{C}$-antilinear part, as we have seen at the beginning of the cours. Let $\Lambda_x^{1,0}$, $\Lambda_x^{0,1}$ denote respectively the space of $\mathbb{C}$-linear and $\mathbb{C}$-antilinear functionals, which are complex vector spaces. One has

$$\Lambda_x^1 = \Lambda_x^{1,0} \oplus \Lambda_x^{0,1}.$$

Set

$$\Lambda_x^{p,q} = (\Lambda_x^{1,0})^{\wedge p} \wedge (\Lambda_x^{0,1})^{\wedge q}.$$

The space of complex-valued $\mathbb{R}$-polylinear skew-symmetric m-forms on T_xM is

$$\Lambda_x^m = \oplus_{p+q=m} \Lambda_x^{p,q}.$$

Definition 8.34 A complex-valued differential m-form ω on M is of *type* (p,q)*,* if its restriction to every tangent space T_xM belongs to the space $\Lambda_x^{p,q}$.

Set

$$\mathcal{E}^m = \mathcal{E}^m(M) = \text{ the space of } C^\infty \ m-\text{forms},$$
$$\mathcal{E}^{p,q} = \text{ the space of } C^\infty \ (p,q)-\text{forms},$$
$$\mathcal{E}^m = \oplus_{p+q=m} \mathcal{E}^{p,q}.$$

The differential $d : \mathcal{E}^m \to \mathcal{E}^{m+1}$ sends each subspace $\mathcal{E}^{p,q}$ to the sum $\mathcal{E}^{p+1,q} \oplus \mathcal{E}^{p,q+1}$. Therefore,

$$d = \partial + \bar{\partial}, \ \partial : \mathcal{E}^{p,q} \to \mathcal{E}^{p+1,q}, \ \bar{\partial} : \mathcal{E}^{p,q} \to \mathcal{E}^{p,q+1}.$$

Example 8.35 Let $f : M \to \mathbb{C}$ be a smooth function, i.e., $f \in \mathcal{E}^0$. Then $\bar{\partial} f$ is the $\mathbb{C}$-antilinear part of the differential *df* introduced at the beginning of the cours. A (p,q)-form on a domain in $\mathbb{C}^n$ is a sum of forms of the type

$$\omega = \sum_{I,J} g_{IJ}(z) dz_I \wedge \overline{dz_J}, \ I = (i_1, \dots, i_p), \ J = (j_1, \dots, j_q), \tag{8.15}$$

$$i_1 < \dots < i_p, \ j_1 < \dots < j_q,$$
$$dz_I = dz_{i_1} \wedge \dots \wedge dz_{i_p}, \ \overline{dz_J} = \overline{dz_{j_1}} \wedge \dots \wedge \overline{dz_{j_q}},$$
$$\bar{\partial}\omega = \sum_{s=1}^n \sum_{I,J} \frac{\bar{\partial}}{\partial \bar{z}_s} g_{IJ}(z) \overline{dz_s} \wedge dz_I \wedge \overline{dz_J}. \tag{8.16}$$

It is well-known that $d^2 = 0$. This implies that

$$\partial^2 = \bar{\partial}^2 = 0,$$

since ∂^2, $\bar{\partial}^2$ are the compositions of the squared differential $d^2 : \mathcal{E}^{p,q} \to \mathcal{E}^{p+q+2}$ with the projections to $\mathcal{E}^{p+2,q}$ and $\mathcal{E}^{p,q+2}$ respectively. Set

$$\mathcal{Z}^{p,q} = \mathrm{Ker}(\bar{\partial}|_{\mathcal{E}^{p,q}}), \ \mathcal{B}p, q = Im(\bar{\partial}|_{\mathcal{E}^{p,q-1}}), \ H^{p,q} = H^{p,q}(M) = \mathcal{Z}^{p,q}/\mathcal{B}^{p,q}.$$

The latter quotient spaces are called the *Dolbeault cohomology* of the manifold M.

Remark 8.36 Every holomorphic mapping of complex manifolds $G : M \to N$ induces the pullback mapping $G^* : H^{p,q}(N) \to H^{p,q}(M)$. The Dolbeault cohomology of biholomorphically equivalent manifolds are isomorphic.

Theorem 8.37 ($\bar{\partial}$-Poincaré Lemma). *For every polydisk $\Delta = \Delta_r \subset \mathbb{C}^n$ one has*

$$H^{p,q} = 0 \text{ whenever } q > 0.$$

Proof The case of dimension one and degree one was treated in Theorem 8.30. Let us prove Theorem 8.37 in the general case. Let $\omega \in \mathcal{E}^{p,q}$, $q > 0$, $\bar{\partial}\omega = 0$. Let us show that there exists a form $\alpha \in \mathcal{E}^{p,q-1}$ such that $\bar{\partial}\alpha = \omega$. This will prove the theorem.

1) Reduction to the case $(0, q)$. The expression (8.15) for the form ω can be written as a sum

$$\omega = \sum_I dz_I \wedge \omega_I, \ \omega_I = \sum_J g_{IJ}\overline{dz_J} \in \mathcal{E}^{0,q},$$

$$\bar{\partial}\omega = (-1)^p \sum_I dz_I \wedge \bar{\partial}\omega_I = 0.$$

The p-forms dz_I are linearly independent. This together with the latter equality implies that $\bar{\partial}\omega_I = 0$. Suppose that we have proved the theorem for $(p, q) = (0, q)$. Then we can find $(0, q-1)$-forms α_I such that $\bar{\partial}\alpha_I = \omega_I$. Set

$$\alpha = (-1)^p \sum_I dz_I \wedge \alpha_I.$$

One has $\bar{\partial}\alpha = \omega$, by definition. This proves the theorem in the general case.

2) Case $(0, q)$: $p = 0$. Then

$$\omega = \sum_J g_J\overline{dz_J}, \ \bar{\partial}\omega = 0.$$

Set

$$K = \cup_{J,\ g_J \not\equiv 0}\{j_1, \dots, j_q\} \subset \{1, \dots, n\},$$
$$|K| = \text{ the cardinality of the set } K;\ |K| \geq q.$$

We prove the existence of a form α such that $\bar{\partial}\alpha = \omega$ by induction in $|K|$.

Induction base: $|K| = q$. Then up to permutation of coordinates we can and will consider that

$$\omega = g(z)\overline{dz_1} \wedge \dots \wedge \overline{dz_q}.$$

The equality $\bar{\partial}\omega = 0$ is equivalent to the statement that the function g is holomorphic in $z_{q+1}, \dots, z_n$. There exists a C^∞ function $f : \Delta \to \mathbb{C}$ holomorphic in the same variables such that

$$\frac{\bar{\partial} f}{\partial z_1} = g,$$

by Theorem 8.30 and its addendum. Set

$$\alpha = f\overline{dz_2} \wedge \dots \wedge \overline{dz_q}.$$

One has $\bar{\partial}\alpha = \omega$, by definition.

Induction step. Let we have proved the existence of the above form α in the case, when $|K| < l$, $l > q$. Let us prove its existence for a form ω with $|K| = l$. Up to permutation of coordinates, we can and will consider that $1 \in K$. Then

$$\omega = \sum_I g_I \overline{dz_1} \wedge \overline{dz_I} + \sum_J g_J \overline{dz_J},\ I = (i_1, \dots, i_{q-1}),\ J = (j_1, \dots, j_q),$$
$$i_s, j_r \in K' = K \setminus \{1\}.$$

The functions g_I are holomorphic in variables z_t, $t \notin K$. This follows from the equality $\bar{\partial}\omega = 0$ and linear independence of the collection of 1-forms $\overline{dz_1} \wedge \overline{dz_I}$ and $\overline{dz_J}$. Therefore, for every I there exists a C^∞ function $f_I : \Delta \to \mathbb{C}$ holomorphic in the same variables such that $\frac{\bar{\partial} f_I}{\partial z_1} = g_I$. Set

$$\beta = \sum_I f_I \overline{dz_I}.$$

The difference $\omega - \bar{\partial}\beta$ is a $\bar{\partial}$-closed form, and it contains only products $\overline{dz_S}$ with $S = (s_1, \dots, s_q)$, $s_j \in K' = K \setminus \{1\}$, by construction. One has $|K'| < l$. Therefore, there exists a form α such that $\bar{\partial}\alpha = \omega - \bar{\partial}\beta$, by the induction hypothesis. Thus,

$$\omega = \bar{\partial}(\alpha + \beta).$$

The induction step is over. The proof of Theorem 8.37 is complete. □

8.5 Holomorphic hypersurfaces. Existence of defining and extended holomorphic functions. Cousin problems

Recall that a holomorphic hypersurface in a complex manifold M is an analytic subset $A \subset M$ such that each its point has a neighborhood U in M where the set $A \cap U$ is defined as zero locus of a holomorphic function on U: a local defining function. Here we study the question of the existence of global defining function.

Definition 8.38 Let $A \subset M$ be a holomorphic hypersurface in a complex manifold M. For every $x \in A$ by $I_A(x)$ we denote the ideal formed by the germs of functions at x holomorphic on its neighborhood in M that vanish on A. We know that it is a principal ideal: it has a generator that divides each element of the ideal. A holomorphic function $F : M \to \mathbb{C}$ is a *global defining function* for the hypersurface A, if $A = \{F = 0\}$ and for every point $x \in A$ the germ of the function F at x generates the corresponding ideal.

Theorem 8.39 *Every holomorphic hypersurface in a polydisk has a global defining function. The same statement holds for hypersurfaces in $\mathbb{C}^n$.*

The proof of the theorem will be postponed till we introduce sheaves. In what follows we will prove its weaker version, under an additional topological assumption.

First of all, let us present an approach to the proof of the theorem via Cousin problems. To do this, we use the following proposition and corollary.

Proposition 8.40 *Let $(A, x) \subset M$ be a germ of holomorphic hypersurface. A germ of holomorphic function f at x generates the ideal $I_A(x)$, if and only if there exists a neighborhood $U = U(x) \subset M$ such that f is holomorphic on U, $A \cap U = \{f = 0\}$ and the differentials $df(y) : T_yM \to \mathbb{C}$ do not vanish on an open and dense subset of points $y \in A \cap U$.*

Proof Let f be a generator of the ideal. Let us prove that its differential does not vanish on the regular part A_{reg} of the germ (A, x), which is open and dense in A. Let $A_1, \dots, A_k$ denote the irreducible components of the germ at x of the hypersurface A. Let h_j denote the irreducible germs at x of holomorphic functions such that $A_j = \{h_j = 0\}$. We already know that

$$f = h_1 \dots h_k$$

up to a unity, see Proposition 5.10 and its proof. Without loss of generality we consider that f is the above product: multiplication by unity does not change non-vanishing of differential at points of the zero locus A. The regular part A_{reg} coincides with the complement of the set A to the singularities of the sets A_j and to their intersections. We already know that $dh_j \neq 0$ on the regular part of the irreducible hypersurface A_j, see Theorem 4.28. This implies that $df \neq 0$ on A_{reg}.

Conversely, let f be a holomorphic function on a neighborhood $U = U(x)$ such that $A \cap U = \{f = 0\}$ and df does not vanish on an open and dense subset in A. The germ at x of the function f belongs to the ideal $I_A(x)$. Hence,

$$f = gh_1 \dots h_k.$$

The germ g is a unity. Indeed, otherwise its nonempty zero locus should lie in A and hence, its irreducible factors are some of h_j. Finally, f is divisible by h_j^2 for some j and hence, $df = 0$ on the irreducible component A_j, – a contradiction. The proposition is proved. □

Corollary 8.41 *Let $A \subset M$ be a holomorphic hypersurface, $x \in A$, f be a generator of the ideal $I_A(x)$. Then there exists a neighborhood $U = U(x) \subset M$ such that f is holomorphic on U and for every $y \in A \cap U$ the germ of the function f at y generates the ideal $I_A(y)$.*

Proof There exists a neighborhood $U = U(x)$ where f is holomorphic and such that the differential df does not vanish on an open and dense subset in $A \cap U$. This together with Proposition 8.40 implies that f generates the ideal $I_A(y)$ for every $y \in A \cap U$ and proves the corollary. □

Let $A \subset M$ be a holomorphic hypersurface. Let us consider its covering by open subsets $U_j \subset M$ such that there exist holomorphic functions $g_j : U_j \to \mathbb{C}$ for which $A \cap U_j = \{g_j = 0\}$ and g_j generate the ideals $I_A y$ for every $y \in A \cap U_j$. Then

$$h_{ij} = \frac{g_j}{g_i} \tag{8.17}$$

are nonvanishing holomorphic functions on $U_i \cap U_j$.

Let us consider the covering of the whole manifold by the above neighborhoods U_j and the complement

$$U_0 = M \setminus A.$$

Set

$$g_0 = 1,\ h_{0j} = \frac{g_j}{g_0} = g_j \text{ on } U_0 \cap U_j. \tag{8.18}$$

Proposition 8.42 *An analytic hypersurface $A \subset M$ admits a global defining function, if and only if there exist nonvanishing holomorphic functions $f_i : U_i \to \mathbb{C}^*$ such that*

$$\frac{f_j}{f_i} = h_{ij}. \tag{8.19}$$

Proof Let F be a defining function. Then the functions $f_j = \frac{g_j}{F}$ are holomorphic and nonvanishing on U_j and satisfy (8.19). Conversely, let $f_j : U_j \to \mathbb{C}^*$ be nonvanishing holomorphic functions satisfying (8.19). Then

$$F = \frac{g_j}{f_j}$$

is a global defining function: the latter fractions are holomorphic on the domains U_j and coincide in their intersections, by (8.19). The proposition is proved. □

Definition 8.43 Let $\mathcal{U}$ be a covering of a manifold M by open subsets U_j. Let $h_{ij} : U_i \cap U_j \to \mathbb{C}^*$ be functions. We say that a collection of the functions h_{ij} is a *multiplicative cocycle,* if $h_{ij} = h_{ji}^{-1}$ and for every indices i, j, k for which $U_i \cap U_j \cap U_k \neq \emptyset$ one has

$$h_{ij} = h_{ji}^{-1}, \ h_{ij}h_{jk}h_{ki} = 1 \text{ on } U_i \cap U_j \cap U_k. \tag{8.20}$$

A collection (h_{ij}) is a *multiplicative coboundary,* if there exist functions $f_j : U_j \to \mathbb{C}^*$ satisfying equation (8.19). A multiplicative cocycle is called C^∞ *(holomorphic)*, if so are h_{ij}.

Remark 8.44 Each coboundary is a cocycle. The converse is not true in general.

Holomorphic (smooth) Multiplicative Cousin Problem. *Given a holomorphic (C^∞) multiplicative cocycle (h_{ij}), find holomorphic (C^∞) nonvanishing functions $f_j : U_j \to \mathbb{C}^*$ satisfying (8.19).*

Corollary 8.45 *Let M be a complex manifold such that each holomorphic Multiplicative Cousin Problem can be solved. Then every holomorphic hypersurface in M admits a global defining function.*

The corollary follows immediately from Proposition 8.42.

Studying the Multiplicative Cousin Problem is closely related to studying the additive one introduced below.

Definition 8.46 Let $\mathcal{U}$ be a covering of a manifold M by open subsets U_j. A collection of functions $h_{ij} : U_i \cap U_j \to \mathbb{C}$ is called an *additive cocycle,* if $h_{ij} = -h_{ji}$ and for every indices i, j, k for which $U_i \cap U_j \cap U_k \neq \emptyset$ one has

$$h_{ij} = -h_{ji}, \;\; h_{ij} + h_{jk} + h_{ki} = 0 \text{ on } U_i \cap U_j \cap U_k. \tag{8.21}$$

A collection (h_{ij}) is an *additive coboundary,* if there exist functions $f_j : U_j \to \mathbb{C}$ such that

$$f_j - f_i = h_{ij} \text{ on } U_i \cap U_j. \tag{8.22}$$

(Each additive coboundary is a cocycle, as in the above remark.) An additive cocycle is called C^∞ *(holomorphic)*, if so are h_{ij}.

Holomorphic (C^∞) Additive Cousin Problem. *Given a holomorphic (C^∞) additive cocycle (h_{ij}), find holomorphic (C^∞) functions $f_j : U_j \to \mathbb{C}$ satisfying (8.22).*

Remark 8.47 An Additive Cousin Problem for a cocycle h_{ij} generates the multiplicative one for the cocycle $e^{h_{ij}}$. If (f_j) is a solution of the additive problem, then the functions e^{f_j} form a solution of the multiplicative one. But the converse is not directly true: taking logarithm transforms a multiplicative cocycle to an additive cocycle *modulo* $2\pi i\mathbb{Z}$.

In what follows first we solve the smooth Additive Cousin Problem for every manifold. Then we show that each holomorphic Additive Cousin Problem on polydisk can be solved. Afterwards we deduce a weak version of Theorem 8.39.

Theorem 8.48 *Every smooth Additive Cousin Problem on every real C^∞-smooth manifold can be always solved.*

In the proof of the theorem we use the following well-known topological fact.

Proposition 8.49 *For every covering of a manifold M by open sets U_j there exists its locally finite covering by open sets V_α such that for every α there exists a $i = i(\alpha)$ such that $\overline{V}_\alpha$ is a compact subset in $U_{i(\alpha)}$. Every locally finite covering V_α admits a partition of unity: there exist C^∞ functions $\rho_\alpha : M \to \mathbb{R}$ with supports in V_α such that $\sum_\alpha \rho_\alpha = 1$.*

Proof Fix a compact exhaustion $K_1 \Subset K_2 \Subset \cdots = M$. Each point $x \in M$ is contained in some U_j. Therefore, we can take its neighborhood $V(x)$ whose closure is a compact subset in U_j. Let us cover the compact K_1 by a finite number the latter neighborhoods $V(x)$. Let W_1 denote their union. The complement $K_2 \setminus W_1$ is a compact set disjoint from the set K_1. Let us cover it by a finite number of neighborhoods $V(x)$ disjoint from the set K_1. Let W_2 denote their union. Then we cover similarly $K_3 \setminus W_2$ etc. The covering of the manifold M thus obtained is locally finite. The existence of partition of unity for a locally finite covering is classical. The proposition is proved. $\square$

Proof of Theorem 8.48. Let M be a manifold, $\mathcal{U}$ be its covering, (h_{ij}) be a C^∞ additive cocycle. Let V_α be a covering from the above proposition, ρ_α be the corresponding partition of unity. Let us introduce the following auxiliary functions $f_{\alpha j}$ on each covering set U_j:

$$f_{\alpha j} = \left[\begin{array}{c} \rho_\alpha h_{ij} \text{ on } V_\alpha \cap U_j \\ 0 \text{ on } U_j \setminus V_\alpha \end{array} \right. ; \quad \text{here } i = i(\alpha) : \overline{V}_\alpha \subset U_i.$$

Set

$$f_j = \sum_\alpha f_{\alpha j}.$$

Claim. *The functions f_j are well-defined and C^∞ on U_j and satisfy (8.22).*

Proof Each function $f_{\alpha j}$ is well-defined and C^∞ on $U_j \cap \overline{V_\alpha}$, by definition and the inclusion $\overline{V}_\alpha \subset U_i$. It is well-defined as a C^∞ function on all of U_j by the formula $f_{\alpha j} = \rho_\alpha h_{ij}$, since both functions coincide on a neighborhood of the latter intersection in U_j and $\rho_\alpha = 0$ outside V_α. Therefore, f_j is well-defined and C^∞ in U_j (local finiteness of the covering by open sets V_α). On the intersection $U_j \cap U_k$ one has

$$f_{\alpha j} - f_{\alpha k} = \rho_\alpha (h_{ij} - h_{ik}) = \rho_\alpha h_{jk},$$

by definition and cocycle identity (8.21). This together with the equality $\sum_\alpha \rho_\alpha = 1$ implies (8.22) and proves the claim. $\square$

The claim implies the statement of the theorem. $\square$

Theorem 8.50 (Cousin). *Every holomorphic Additive Cousin Problem on each polydisk is solvable.*

Proof Let $\Delta \subset \mathbb{C}^n$ be a polydisk, $\mathcal{U}$ be its covering, (h_{ij}) be a holomorphic additive cocycle. Let (h_j), $h_j : U_j \to \mathbb{C}$ be a C^∞ solution of the corresponding Additive Cousin Problem, $h_j - h_i = h_{ij}$, which exists by Theorem 8.48.

The $(0,1)$-forms $\bar{\partial} h_j$ on U_j are $\bar{\partial}$-closed and coincide on the intersections of the covering sets:

$$\bar{\partial} h_j - \bar{\partial} h_i = \bar{\partial} h_{ij} = 0 \text{ on } U_i \cap U_j,$$

since the functions h_{ij} are holomorphic. Therefore, there exists a global $\bar{\partial}$-closed C^∞ $(0,1)$-form ω on Δ coinciding with $\bar{\partial} h_j$ on U_j for every j. Let $f : \Delta \to \mathbb{C}$ be a C^∞-solution of the $\bar{\partial}$-problem: $\bar{\partial} f = \omega$. It exists by Theorem 8.37. The functions

$$f_j = h_j - f$$

are holomorphic on U_j and satisfy (8.22), by construction. Therefore, they present a solution of the holomorphic Cousin Problem. This proves the theorem. $\square$

Definition 8.51 Let M be a complex manifold, $A \subset M$ be a holomorphic hypersurface. A continuous function $G : M \to \mathbb{C}$ is said to be C^0-*defining function* of the hypersurface A, if $A = \{G = 0\}$ and the following condition holds. Let $U \subset M$ be an open subset intersecting A, $f : U \to \mathbb{C}$ be a holomorphic function such that $A \cap U = \{f = 0\}$ and f generates the ideal $I_A(y)$ for every $y \in A \cap U$. Then $\frac{f}{G}$ is a nonvanishing continuous function on U.

Theorem 8.52 *Let $\Delta \subset \mathbb{C}^n$ be a polydisk, $A \subset \Delta$ be a holomorphic hypersurface that admits a C^0-defining function. Then it admits a holomorphic global defining function.*

Proof Let us consider the Multiplicative Cousin Problem associated to A: let a covering of the polydisk, g_j and h_{ij} be the same, as in (8.17) and (8.18). We pass to a covering by smaller simply connected sets, which we will denote U_j. Set

$$h_j = \frac{g_j}{G}.$$

The functions h_j are continuous nonvanishing on U_j. Let H_j denote continuous branches of their logarithms on U_j. The differences

$$u_{ij} = H_j - H_i$$

are holomorphic in $U_i \cap U_j$, being continuous branches of logarithms of holomorphic functions h_{ij}. They satisfy the cocycle identity $u_{ij} + u_{jk} + u_{ki} = 0$ on triple intersections, by construction. Let $v_j : U_j \to \mathbb{C}$ denote the solution of the corresponding holomorphic Additive Cousin Problem: $v_j - v_i = u_{ij}$. The functions

$$f_j = e^{v_j}$$

form a solution of the holomorphic Multiplicative Cousin Problem for the initial cocycle h_{ij}. This together with Proposition 8.42 proves the theorem. □

Definition 8.53 Let M be a complex manifold, $A \subset M$ be an analytic subset. We say that a function $g : A \to \mathbb{C}$ is *holomorphic,* if it is *locally* the restriction to A of a holomorphic function. That is, for every $x \in A$ there exist a neighborhood $U = U(x) \subset M$ and a holomorphic function $G : U \to \mathbb{C}$ such that $g = G$ on $A \cap U$.

Theorem 8.54 *Let $\Delta \subset \mathbb{C}^n$ be a polydisk. Let $A \subset \Delta$ be a hypersurface that admits a global defining holomorphic function*

$$f : \Delta \to \mathbb{C},\ A = \{f = 0\};\ f \text{ generates the ideals } I_A(x),\ x \in A.$$

Then every holomorphic function $g : A \to \mathbb{C}$ is the restriction to A of a global holomorphic function $G : \Delta \to \mathbb{C}$.

Proof Let us reduce the theorem to an additive Cousin problem. Consider a covering of the set A by open subsets U_j in Δ such that on each U_j there exists a holomorphic function $g_j : U_j \to \mathbb{C}$ for which $g_j|_{A \cap U_j} = g$. Set

$$U_0 = \Delta \setminus A,\ g_0 = 0.$$

One has $g_j - g_i = 0$ on $A \cap U_i \cap U_j$. Therefore,

$$g_j - g_i = f h_{ij} \text{ on } U_i \cap U_j,\ h_{ij} : U_i \cap U_j \to \mathbb{C} \text{ is a holomorphic function,}$$

since f generates the ideal $I_A(x)$ for every $x \in A$. The collection of functions h_{ij} is a cocycle for holomorphic Additive Cousin Problem. Let h_j denote its solution: a collection of holomorphic functions $h_j : U_j \to \mathbb{C}$ such that $h_j - h_i = h_{ij}$ on $U_i \cap U_j$. One has

$$g_j - g_i = f h_{ij} = f h_j - f h_i,$$

by construction. Therefore,

$$G = g_j - f h_j = g_i - f h_i \text{ on } U_i \cap U_j,$$

and G is a global holomorphic function on Δ. One has $G|_A = g$, since $g_j - f h_j = g_j = g$ on $A \cap U_j$, by definition. This proves the theorem. □

8.6 Sheaves and cohomology. Relation to Cousin problems and singular cohomology

Definition 8.55 Let D be a topological space. A *sheaf of Abelian groups* over the space D (called *base*) is a topological space $\mathcal{J}$ equipped with a projection $\pi : \mathcal{J} \to D$ satisfying the following conditions:

1) The projection π is a local homeomorphism: each point $x \in \mathcal{J}$ has a neighborhood in $\mathcal{J}$ that is mapped by π homeomorphically onto a neighborhood of the image $\pi(x)$ in D;

2) for every $x \in D$ the preimage $\pi^{-1}(x)$ has a structure of Abelian group;

3) the group structure is continuous.

Remark 8.56 Let us explain the above condition 3) in more detail. Set

$$\mathcal{J} \circ \mathcal{J} = \{(x_1, x_2) \in \mathcal{J} \times \mathcal{J} \mid \pi(x_1) = \pi(x_2)\} = (\pi \times \pi)^{-1}(\text{diagonal} \subset D \times D).$$

Condition 3) is equivalent to the condition saying that the mapping

$$\mathcal{J} \circ \mathcal{J} \to \mathcal{J} : \ (x_1, x_2) \mapsto x_1 - x_2$$

is continuous.

Example 8.57 Let D be a manifold. Let $\mathcal{J}$ denote the space of germs of functions (differential forms, more generally, sections of a vector bundle) from a given class $\mathcal{R}$ from the following list.

1) $\mathcal{R} = \mathbb{Z}, \mathbb{R}, \mathbb{C}$: this is the space of germs of *locally constant* functions with values in $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ respectively;

2) D is a complex manifold and $\mathcal{R} = \mathcal{O}$ is the space of germs of holomorphic functions (forms, sections);

3) $\mathcal{R}$ is the space of germs of C^∞ functions (forms, sections).

On each one of the above spaces of germs we introduce the following topology, which coincides with (8.4) in the case of holomorphic functions. We treat the case of functions: the cases of forms (sections of vector bundles) are treated analogously. Let (g, x) be a germ of function from class $\mathcal{R}$ at a point $x \in D$ defined on its neighborhood $U = U(x) \subset D$. Set

$$W_U(g, x) = \{(g, y) \mid y \in U\}. \tag{8.23}$$

The sets $W_U(g, x)$ form a basis of topology on the space of germs.

Remark 8.58 The spaces 1) and 2) of locally constant and holomorphic germs are Hausdorff, by Proposition 8.26, while space 3) of smooth germs isn't. Indeed, let f_1, f_2 denote two arbitrary germs of smooth functions of one variable at the origin that coincide on the left interval $x < 0$ and differ on the right interval $x > 0$. Then $(f_1, 0) \neq (f_2, 0)$, but $(f_1, y) = (f_2, y)$ for every $y < 0$. Therefore, every two neighborhoods (8.23) of germs $(f_1, 0)$ and $(f_2, 0)$ intersect. Thus, the topology is not Hausdorff.

Definition 8.59 A *section* of a sheaf $\mathcal{J}$ over a subset U in the base is a continuous inverse $\pi^{-1} : U \to \mathcal{J}$, if it exists. A section over the whole base is called a *global section.* The space of sections over a subset U is an Abelian group, which we will denote $\mathcal{J}_U$. They are global sections of the preimage $\pi^{-1}(U)$, which is a sheaf that is called the *restriction* to U of the sheaf $\mathcal{J}$.

In all our examples the sheaf under question is always defined by the collection of its sections over open subsets: the so-called pre-sheaf, see the following definition.

Definition 8.60 Let D be a topological space, and let $\mathcal{U}$ be a base of topology on D. A *pre-sheaf* of Abelian groups on D is a collection of Abelian groups $(\mathcal{J}_U)_{U \in \mathcal{U}}$ and homomorphisms $\mathcal{J}_V \to \mathcal{J}_U$ for $U \subset V$ called *restriction homomorphisms* that satisfy the following property: for every three subsets $U \subset V \subset W$ one has

$$\rho_{UV} \circ \rho_{VW} = \rho_{UW}.$$

Remark 8.61 The sections of a sheaf over basic open sets form a pre-sheaf. Conversely, every pre-sheaf $\mathcal{J}_U$ induces a sheaf $\mathcal{J}$ as follows. For every $z \in D$ let $\mathcal{U}_z$ denote the collection of basic open sets containing z. Set

$$\mathcal{J}_z = \lim_{U \in \mathcal{U}_z} \operatorname{ind} \mathcal{J}_U, \;\; \mathcal{J} = \cup_{z \in D} \mathcal{J}_z.$$

In more detail, $\mathcal{J}_z$ is the union of all the Abelian groups $\mathcal{J}_U$, $U \in \mathcal{U}_z$ quotiented by the following equivalence relation: $g_1 \in \mathcal{J}_{U_1}$ is said to be equivalent to $g_2 \in \mathcal{J}_{U_2}$, we write

$$g_1 \simeq_z g_2,$$

if there exists a $V \in \mathcal{U}_z$, $V \subset U_1 \cap U_2$ such that $\rho_{VU_1} g_1 = \rho_{VU_2} g_2$. The spaces $\mathcal{J}_z$ inherit a natural Abelian group structure, and their union $\mathcal{J}$ equipped with the natural topology is a sheaf over D. Namely, for every $U \in \mathcal{U}$ and $z \in U$ consider the natural projection

$$\mathcal{J}_U \to \mathcal{J}_z : \;\; g \mapsto [g]_z = g / \simeq_z .$$

The sets

$$W_U(g) = \{[g]_z \mid z \in U\}$$

form a basis of topology on $\mathcal{J}$. This is the topology of the sheaf $\mathcal{J}$: the local homeomorphicity of the projection $[g]_z \mapsto z$ and continuity of group structure follow from definition.

For every $U \in \mathcal{U}$ there is a natural homomorphism of the pre-sheaf groups to the corresponding section spaces,

$$\psi_U : \mathcal{J}_U \to \Gamma(U) :$$

for every $g \in \mathcal{J}_U$ the image $\psi_U(g)$ is the section sending $z \in U$ to $[g]_z$.

Remark 8.62 In general, the latter homomorphisms ψ_U are not isomorphisms. But for all the sheaves listed above (germs of functions, forms, sections...) the pre-sheaves of their sections over basic open sets induce the corresponding sheaves via the above construction; the corresponding homeomorphisms ψ_U are isomorphisms by definition.

Definition 8.63 A *complex of Abelian groups* is a sequence of Abelian groups C^j, $j \geq 0$ with a mapping $\delta : C^j \to C^{j+1}$ called the *differential,*

$$0 \to C^0 \to C^1 \to C^2 \dots, \quad \delta^2 = 0.$$

The elements of the group C^j are called *cochaines.* The *j-th cohomology* of the complex is the quotient Abelian group

$$H^j = H^j_C = \mathrm{Ker}(\delta|_{C^j})/Im(\delta|_{C^{j-1}}).$$

The elements of the above kernel are called *cocycles*; the elements of the latter image are called *coboundaries.*

In what follows we recall the construction of Čech complex and cohomology associated to a sheaf. To do that, we will need the following background material from the homological algebra. A *homomorphism* of two complexes C^j and E^j is a sequence of homomorphisms $f : C^j \to E^j$ that forms a commutative diagram with the differentials. Every homomorphism of complexes induces homomorphisms in the cohomology:

$$f_* : H^j_C \to H^j_E.$$

Definition 8.64 Two homomorphisms of complexes $f, g : C^j \to E^j$ are *chain-homotopic,* if there exists a sequence of homomorphisms $\theta_j : C^j \to E^{j-1}$ such that

$$\delta \circ \theta_j + \theta_{j+1} \circ \delta = f - g \text{ for all } j. \tag{8.24}$$

Proposition 8.65 *Every two chain-homotopic homomorphisms induce the same homomorphism in the cohomology.*

Proof Let $f, g : C^j \to E^j$ be chain-homotopic. Then for every $c \in \operatorname{Ker}(\delta|_{C^j})$ one has

$$f(c) - g(c) = \delta(\theta_j(c)) + \theta_{j+1}(\delta c) = \delta(\theta_j(c)),$$

since $\delta c = 0$ by definition. Thus, the above difference is a coboundary, and hence, represents zero element in the cohomology. This proves the proposition. $\square$

Let $\mathcal{J}$ be a sheaf of Abelian groups over a topological space D. The construction of Čech cohomology associates an Abelian group complex to each covering $\mathcal{U}$ of the base D by open sets U_j. Set

$$C^k = \oplus_{U_{i_0}\cap\dots\cap U_{i_k}\neq\emptyset;\ i_s\neq i_r \text{ for } s\neq r}\Gamma(U_{i_0}\cap\dots\cap U_{i_k}) :$$

Antisymmetry numeration convention. The direct sum C^k is an Abelian group whose elements are collections $(h_{i_0\dots i_k})$ indexed by those unordered sets $\{i_0, \dots, i_k\}$ of distinct indices, for which the above intersections are non-empty. In what follows, we consider sections $h_{i_0\dots i_k} \in \Gamma(U_{i_0} \cap \dots \cap U_{i_k})$ numerated by *ordered* collections $I = (i_0, \dots, i_k)$: for every I we take the corresponding section h_I so that

$$h_{\sigma I} = (-1)^{\operatorname{sign}\sigma} h_I \text{ for every permutation } \sigma \in S_{k+1}. \tag{8.25}$$

Definition 8.66 A collection h of sections $h_{i_0\dots i_k} \in \Gamma(U_{i_0} \cap \dots \cap U_{i_k})$ numerated by ordered indices $I = (i_0, \dots, i_k)$ and satisfying antisymmetry equation (8.25) is called a *k-cochain.*

Remark 8.67 The space of k-cochains is an Abelian group isomorphic to C^k. In what follows *the symbol C^k will always denote the space of k-cochains.*

The differential $\delta : C^k \to C^{k+1}$ is defined as follows: for every $h \in C^k$ set

$$(\delta h)_{i_0\dots i_{k+1}} = \sum_{s=0}^{k+1} (-1)^s h_{i_0\dots\hat{i}_s\dots i_{k+1}}|_{U_{i_0}\cap\dots U_{i_{k+1}}}, \tag{8.26}$$

where $\hat{i}_s$ means that we take all the indices i_l except for i_s, i.e., with $l = 1, \dots, s-1, s+1, \dots, k+1$. One has

$$\delta^2 = 0.$$

Indeed, for every $h \in C^{k-1}$ one has

$$(\delta^2 h)_{i_0 \dots i_{k+1}} = \sum_{s=0}^{k+1} (-1)^s (\delta h)_{i_0 \dots \hat{i}_s \dots i_{k+1}}$$

$$= \sum_{s=0}^{k+1} (\sum_{l<s} (-1)^{s+l} h_{i_0 \dots \hat{i}_l i_{l+1} \dots \hat{i}_s i_{s+1} \dots i_{k+1}} + \sum_{l>s} (-1)^{l+s-1} h_{i_0 \dots \hat{i}_s i_{s+1} \dots \hat{i}_l i_{l+1} \dots i_{k+1}}.$$

The latter sum with $l > s$ equals the former sum with $l < s$, and they are taken with opposite signs. Therefore, they cancel out and $\delta^2 = 0$.

Definition 8.68 Let $\mathcal{J}$, $D, \mathcal{U}$ and $\delta : C^k \to C^{k+1}$ be as above:

$$0 \to C^0 \to C^2 \to C^2 \dots.$$

The cohomology groups

$$H^k = H^k_{\mathcal{U}}(D, \mathcal{J}) = \mathrm{Ker}(\delta|_{C^k})/Im(\delta|_{C^{k-1}})$$

are called the *Čech cohomology* associated to the covering $\mathcal{U}$ with the coefficients in the sheaf $\mathcal{J}$.

Example 8.69 A collection $h = (h_j) \in C^0$ is a cocycle, if and only the sections $h_j : U_j \to \mathcal{J}$ coincide on the intersections $U_i \cap U_j$ and hence, define a global section $D \to \mathcal{J}$. This implies that

$$H^0_{\mathcal{U}}(D, \mathcal{J}) = \Gamma(D).$$

A collection $(h_{ij}) \in C^1$ is a cocycle, if and only if it is a cocycle in the sense of the Cousin Problem:

$$h_{ij} + h_{jk} + h_{ki} = 0 \text{ on } U_i \cap U_j \cap U_k.$$

A cocycle (h_{ij}) is a coboundary, if and only if the corresponding Cousin Problem can be solved: there exists a collection of sections $h_s : U_s \to \mathcal{J}$ such that $h_j - h_i = h_{ij}$. Therefore, *the non-vanishing of the first Čech cohomology class of a given cocycle in C^1 represents an obstruction for solving the corresponding Cousin Problem.*

Definition 8.70 Let $\mathcal{V}$, $\mathcal{U}$ be two coverings of a topological space D by open subsets $V_\alpha, U_j \subset D$ respectively. We say that $\mathcal{V} < \mathcal{U}$, if for every $V_\alpha \in \mathcal{V}$ there exists a covering element $U_{\phi(\alpha)} \in \mathcal{U}$ such that $V_\alpha \subset U_{\phi(\alpha)}$. (The correspondence $\alpha \mapsto \phi(\alpha)$ may be not unique.)

For every index correspondence $\phi : \alpha \mapsto \phi(\alpha)$ as above we define the following mapping between cochain spaces

$$\rho_\phi : C^k_{\mathcal{U}} \to C^k_{\mathcal{V}} : \ (\rho_\phi h)_{\alpha_0 \dots \alpha_k} = h_{\phi(\alpha_0)\dots\phi(\alpha_k)}. \tag{8.27}$$

The mappings ρ_ϕ commute with differentials and form a mapping of cochain complexes. Thus, they induce the mappings

$$(\rho_\phi)_* : H^k_{\mathcal{U}} \to H^k_{\mathcal{V}}.$$

Proposition 8.71 (Task 5, part 2, Problem 1). *For every two coverings $\mathcal{V} < \mathcal{U}$ and two index correspondences ϕ_1 and ϕ_2 as in the above definition the cochain complex mappings $\rho_{\phi_1}, \rho_{\phi_2} : C^k_{\mathcal{U}} \to C^k_{\mathcal{V}}$ are chain-homotopic: there exist homomorphisms $\theta_k : C^k_{\mathcal{U}} \to C^{k-1}_{\mathcal{V}}$ such that*

$$\rho_{\phi_2} - \rho_{\phi_1} = \delta\theta_k + \theta_{k+1}\delta. \tag{8.28}$$

Equality (8.28) holds for homomorphisms

$$\theta_l : C^l_{\mathcal{U}} \to C^{l-1}_{\mathcal{V}}, \ (\theta_l h)_{\alpha_0\dots\alpha_{l-1}} = \sum_{s=0}^{l-1} (-1)^s h_{\phi_1(\alpha_0)\dots\phi_1(\alpha_s)\phi_2(\alpha_s)\dots\phi_2(\alpha_{l-1})}.$$

Corollary 8.72 *For every coverings $\mathcal{V} < \mathcal{U}$ the induced homomorphism $(\rho_\phi)_* : H^k_{\mathcal{U}} \to H^k_{\mathcal{V}}$ does not depend on the choice of index mapping ϕ.*

The corollary follows from Propositions 8.71 and 8.65.

Now let us define the covering-independent Čech cohomology

$$H^k(D, \mathcal{J}) = \lim_{\mathcal{U}} H^k_{\mathcal{U}}(D, \mathcal{J}).$$

Here we take the inductive limit under passing to smaller and smaller coverings. In more detail,

$$H^k(D, \mathcal{J}) = (\oplus_{\mathcal{U}} H^k_{\mathcal{U}}(D, \mathcal{J}))/ \simeq :$$

by definition, $c_1 \in H^k_{\mathcal{U}}$ is equivalent to $c_2 \in H^k_{\mathcal{V}}$, if there exists a smaller covering $\mathcal{W} < \mathcal{U}, \mathcal{V}$ with the following property. Let $\phi : \mathcal{W} \to \mathcal{U}$, $\psi : \mathcal{W} \to \mathcal{V}$ denote the corresponding index mappings such that $W_\alpha \subset U_{\phi(\alpha)}, V_{\psi(\alpha)}$. Then $(\rho_\phi)_*(c_1) = (\rho_\psi)_* c_2$.

Remark 8.73 One has $H^k(D, \mathcal{J}) = 0$, if and only if for every covering $\mathcal{U}$ and every $c \in H^k_{\mathcal{U}}(D, \mathcal{J})$ there exists a smaller covering $\mathcal{V} < \mathcal{U}$, let $\phi : \mathcal{U} \to \mathcal{V}$ denote the corresponding inclusion mapping, such that $(\rho_\phi)^* c = 0$ in $H^k_{\mathcal{V}}$. Given a covering $\mathcal{U}$, one has a natural homomorphic projection

$$\rho : H^k_{\mathcal{U}}(D, \mathcal{J}) \to H^k(D, \mathcal{J}). \tag{8.29}$$

Proposition 8.74 *The latter projection is an isomorphism, if for every other covering $\mathcal{V}$ there exists a smaller covering $\mathcal{W} < \mathcal{U}, \mathcal{V}$ such that the mapping $\rho_\phi : H^k_{\mathcal{U}}(D, \mathcal{J}) \to H^k_{\mathcal{W}}(D, \mathcal{J})$ is an isomorphism.*

The proposition follows from definition and the above remark.

Theorem 8.75 *Let M be a real manifold, $n = dim M$. Let $\mathcal{J} = \mathbb{Z}$ be the sheaf of locally constant integer-valued functions. Then the cohomology groups $H^k(M, \mathbb{Z})$ are isomorphic to the corresponding singular cohomology groups denoted by the same symbols.*

The theorem is proved below. To do this, fix an arbitrary triangulation T of the manifold M. Every simplex of the triangulation T (of any dimension k) will be identified with the collection of vertices $A_0, \dots A_k$. Let us consider the simplicial complex

$$0 \to C^1_T \to C^2_T \to \dots, \ C^k_T = \oplus_{\text{simplexes } A_0 \dots A_k} \mathbb{Z}.$$

Thus, each element of the group C^k_T is a collection of integer numbers indexed by unordered collections of $k+1$ distinct vertices of the triangulation T that are vertices of the same simplex. Let us consider the collections of integer numbers $h_{A_0 \dots A_k}$ numerated by *ordered* vertex collections $A = (A_0, \dots, A_k)$ satisfying the antisymmetry relation

$$h_{\sigma A} = (-1)^\sigma h_A \text{ for every } \sigma \in S_{k+1}.$$

A collection of integers h_A satisfying the latter antisymmetry relation is called a *simplicial k-cochain.* The k-cochains form an Abelian group isomorphic to C^k_T. From now on by C^k_T we will denote the group of k-cochains associated to the triangulation T. The differential $\delta : C^k \to C^{k+1}$ acts as follows. Let $c \in C^k$, and let $A_0 \dots A_k$ be a simplex of the triangulation T (with ordered vertices). Set

$$(\delta c)_{A_0 \dots A_{k+1}} = \sum_{s=0}^{k+1} (-1)^j c_{A_0 \dots \hat{A}_s \dots A_{k+1}}.$$

One has $\delta^2 = 0$, as in the case of sheaf cohomology (the same calculation). The quotients

$$H^k_T = \mathrm{Ker}(\delta|_{C^k_T}) / (\mathrm{Im}(\delta|_{C^{k-1}_T})$$

are called the *cohomology groups of the simplicial complex,* or briefly, the *simplicial cohomology.*

Theorem 8.76 *The simplicial cohomology of a triangulated manifold is naturally isomorphic to its (triangulation independent) singular cohomology.*

This is a classical theorem from topology. We will use it as a given theorem and will not prove it here.

For each vertex v of the triangulation let St_v denote the union of the adjacent simplexes of higher dimension: the latter union is an open subset in M called the *star* at the point v. The stars form a covering of the manifold M, which will be denoted by $St(T)$.

Proposition 8.77 *There exist natural isomorphisms*

$$C_T^k \simeq C_{St(T)}^k(M,\mathbb{Z}),\ \ H_T^k \simeq H_{St(T)}^k(M,\mathbb{Z}). \tag{8.30}$$

Proof For every vertices $v_0,\dots,v_k$ of the triangulation T the corresponding stars intersect, if and only if the vertices form a simplex of the triangulation. This implies that each k-cochain with integer coefficients can be considered as a collection of integers $h_{v_0\dots v_k}$ indexed by k-simplexes with ordered vertices $v_0,\dots,v_k$. It can be obviously be viewed as a simplicial k-cochain, and vice versa. This yields an isomorphism $C^kT \simeq C_{St(T)}^k(M,\mathbb{Z})$, which obviously commutes with the differential and induces an isomorphism in the cohomology. The proposition is proved. □

Proposition 8.78 *Let T_1, T_2 be two triangulations of the manifold M so that T_2 is obtained by subdivision of some simplexes of the triangulation T_1: we will briefly write $T_2 < T_1$. Then $St(T_2) < St(T_1)$. The natural homomorphisms*

$$\psi : C_{T_1}^k \to C_{T_2}^k,\ \rho_\phi : C_{St(T_1)}^k \to C_{St(T_2)}^k$$

defined by restrictions from simplexes of the triangulation T_1 to smaller simplexes of the triangulation T_2 form a commutative diagram with the first isomorphism from (8.30). They induce isomorphisms in the cohomology groups.

Proof The commutation with the isomorphism (8.30) follows from definition. The above natural mapping $\psi : C_{T_1}^k \to C_{T_2}^k$ induces an isomorphism in the cohomology (a classical theorem). This together with the previous statement implies that the corresponding mapping $\rho_\phi : C_{St(T_1)}^k \to C_{St(T_2)}^k$ also induces an isomorphism in the cohomology. This proves the proposition. □

Proposition 8.79 *For every covering $\mathcal{U}$ of a manifold by open sets and every given triangulation T there exists another triangulation obtained by its subdivision whose star covering is inscribed into $\mathcal{U}$.*

Proof Let M be the underlying manifold. Fix some its triangulation T. Consider a compact exhaustion $K_1 \Subset K_2 \Subset \cdots = M$ whose elements K_j are unions of closures of star neighborhoods of vertices. The set K_2 can be covered by a finite number of sets $U_j \in \mathcal{U}$. Therefore, one can subdivide the simplexes in K_2 barycentrically so that for every vertex of the new triangulation in the set K_1 its star neighborhood is contained in some U_j. We extend the triangulation of the set K_2 thus obtained to a triangulation $T' < T$ of the manifold M so that $T = T'$ on $\overline{M \setminus K_3}$. To do this, it suffices to do the following subdivision of each higher dimension simplex $S = A_0 \dots A_k$ in $K_3 \setminus Int(K_2)$ adjacent to K_2. Let, say, the face $F_k = A_0 \dots A_{k-1}$ be not contained in K_2. Fix a point O in the interior of the simplex $A_0 \dots A_k$. Let us first subdivide S by the simplexes $S_j = A_0 \dots A_{j-1} O A_{j+1} \dots A_k$ with the vertex O and opposite face coinciding with a face of the ambient simplex S. For every face $F_j \subset K_2$. let us subdivide the corresponding simplex S_j by simplexes with vertex O according to the subdivision of the face F_j induced by the subdivision of the adjacent simplex in K_1. The key property of this construction is that *the faces $F_s \not\subset K_2$ are not subdivided.* This implies that the union of the simplexes contained in K_2 and those simplexes that are adjacent to K_2 and subdivided as above and the other simplexes of the initial triangulation T forms a triangulation of the manifold M coinciding with T on $\overline{M \setminus K_3}$. Afterwards we subdivide new simplexes in $K_3 \setminus K_2$ barycentrically to achieve that the star neighborhoods of the vertices in $K_3 \setminus K_2$ be contained in some domains U_j. We extend the subdivision thus obtained to a global triangulation via the above construction; then *the triangulation of the set K_1 remains unchanged.* Then we apply the same procedure for the complement $K_4 \setminus K_3$ (the triangulation on K_2 remains unchanged) etc. The triangulation thus obtained is the one we are looking for. $\square$

Proof of Theorem 8.75. Fix a triangulation T_1 of the manifold M. Let us show that the projection $\rho : H^k_{St(T_1)}(M, \mathcal{J}) \to H^k(M, \mathcal{J})$ is an isomorphism. Fix an arbitrary covering $\mathcal{U} < St(T_1)$. Let $T_2 < T$ be a smaller triangulation such that $St(T_2) < \mathcal{U}$: it exists by the above proposition. The inscription homomorphism $H^k_{St(T_1)} \to H^k_{St(T_2)}$ is an isomorphism, by Proposition 8.78. This together with Proposition 8.74 implies the statement of Theorem 8.75. $\square$

8.7 Exact sequences. Corollaries: calculations; existence of global defining functions

Definition 8.80 Let A and B be sheaves over the same base D with projections π_A and π_B respectively. A *homomorphism* $h : A \to B$ is a continuous mapping of the total spaces of sheaves that commutes with the projection and yields a group homomorphism $h_x : \pi_A^{-1}(x) \to \pi_B^{-1}(x)$ of the projection preimages. If its inverse is a sheaf homomorphism, then it is called an isomorphism.

Remark 8.81 Every sheaf homomorphism induces a homomorphism of the groups of sections over every subset in the base.

Remark 8.82 Let $h : A \to B$ be a sheaf homomorphism. Let $\pi_A : A \to D$, $\pi_B : B \to D$ be sheaf projections. For every $x \in D$ let $K_x \subset \pi_A^{-1}(x)$ denote the kernel of the homomorphism h_x. The union of the groups h_x considered as a subset in the total space of the sheaf A is a sheaf called the *kernel sheaf* of the homomorphism h. (Exercise: prove this.)

Definition 8.83 An *exact sequence* of Abelian groups is a sequence of homomorphisms

$$f_j : A^j \to A^{j+1};\ A^1 \to A^2 \to A^3 \to \dots$$

such that $f_{j+1} \circ f_j = 0$ and $\operatorname{Ker} f_{j+1} = Imf_j$. A *short exact sequence* is an exact sequence of the type

$$0 \to A \to B \to C \to 0;\ f : A \to B,\ g : B \to C. \tag{8.31}$$

Its shortness is equivalent to the conditions saying that

$$f \text{ is injective, } g \text{ is surjective, } Imf = \operatorname{Ker} g,\ C \simeq B/ImA.$$

A (short) exact sequence of sheaves of Abelian groups is defined analogously.

In what follows we consider short exact sequences (8.31) of Abelian group complexes

$$0 \to A^1 \to A^2 \to \dots;\ 0 \to B^1 \to B^2 \to \dots;\ 0 \to C^1 \to C^2 \to \dots.$$

The corresponding differentials $\delta : A^j \to A^{j+1}$ will be denoted by the same symbol δ. The short exact sequences $0 \to A^k \to B^k \to C^k \to 0$ should commute with the differentials.

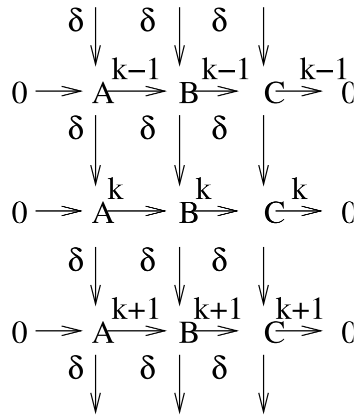

Figure 5: A short exact sequence of Abelian group complexes.

Theorem 8.84 *Every short exact sequence of Abelian group complexes induces a long exact sequence in the cohomology:*

$$0 \to H^0(A) \to H^0(B) \to H^0(C) \to H^1(A) \to H^1(B) \to H^1(C) \to H^2(A) \to \dots, \tag{8.32}$$

where the homomorphisms $H^k(A) \to H^k(B)$, $H^k(B) \to H^k(C)$ are the homomorphisms f_*, g_* induced by the short sequence homomorphisms $f : A^k \to B^k$, $g : B^k \to C^k$.

Addendum. *The homomorphisms $\delta^* : H^k(C) \to H^{k+1}(A)$ from the above sequence are defined as follows. Let $c \in C^k$ be a cocycle. Fix its preimage $b \in g^{-1}(c)$. Then $g(\delta(b)) = \delta c = 0$, by commutativity of the diagram. Therefore, $\delta(b) \in f(A^{k+1})$, by exactness. That is, there exists an $a \in A^{k+1}$ such that $f(a) = \delta b$. The correspondence $c \mapsto a$ induces a homomorphism $\delta^* : H^k(C) \to H^{k+1}(A)$ that forms an exact sequence (8.32) together with f_* and g_*.*

The verification of well-definedness of the above homomorphism δ^* and exactness of the sequence (8.32) is straightforward: everything follows from the commutative diagram in the figure and the exactness of its horizontal lines. Here we will check only the well-definedness of δ^*.

Claim 1. *For every $c \in C^k$ with $\delta c = 0$ the corresponding $a \in A^{k+1}$ is a cocycle well-defined up to a coboundary.*

Proof One has $f(\delta a) = \delta(\delta b) = 0$, by commutativity. Hence, $\delta a = 0$, by the injectivity of the homomorphism f. The element b is well-defined up to addition of an element $q = \operatorname{Ker} g = Imf$. One has $q = f(\alpha)$ for a certain

$\alpha \in A^k$. This together with commutativity implies that $\delta q = f(\delta\alpha)$. Hence, the cocycle a is uniquely defined up to addition of a coboundary $\delta\alpha$. $\square$

Claim 2. *The cohomology class $[a]$ depends only on the cohomology class $[c]$.*

Proof It suffices to show that a is a coboundary, whenever c is a coboundary. Let $c = \delta\gamma$. Choose a $\beta \in g^{-1}(\gamma)$, set $b = \delta\gamma$: $g(b) = c$. One has $\delta b = 0$. Hence, the construction from the addendum yields $a = 0$. $\square$

Corollary 8.85 *Every short exact sequence*

$$0 \to A \to B \to C \to 0$$

of sheaves of Abelian groups over the same base induces a long exact sequence of the cohomology groups

$$0 \to H^0(D, A) \to H^0(D, B) \to H^0(D, C) \to H^1(D, A) \to H^1(D, B) \to H^1(D, C) \to H^2(D, A) \to \dots \tag{8.33}$$

The corollary is a direct application of the above theorem to the corresponding Čech complexes for every covering.

Example 8.86 Let M be a complex manifold. Consider the sheaves $\mathbb{Z}$, $\mathcal{O}$, $\mathcal{O}^*$ of respectively locally-constant integer-valued, holomorphic and holomorphic non-vanishing functions. One has a natural short exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0, \tag{8.34}$$

where the homomorphism $\mathbb{Z} \to \mathcal{O}$ is the natural inclusion, and $\mathcal{O} \to \mathcal{O}^*$ sends a holomorphic function $h(z)$ to its exponent $e^{h(z)}$. In the case, when $M = \Delta$ is a polydisk, one has $H^k(\Delta, \mathbb{Z}) = 0$ for $k > 0$, by Theorem 8.75 and contractibility of the polydisk (which implies triviality of its singular cohomology). Thus, the corresponding long exact cohomology sequence takes the form

$$\dots \to 0 \to H^k(\Delta, \mathcal{O}) \to H^k(\Delta, \mathcal{O}^*) \to 0 \to \dots, \ k \geq 1.$$

This implies that

$$H^k(\Delta, \mathcal{O}) \simeq H^k(\Delta, \mathcal{O}^*) \text{ for every } k \geq 1. \tag{8.35}$$

On the other hand, one has

$$H^1(\Delta, \mathcal{O}) = 0,$$

by Cousin Theorem 8.50. This implies that

$$H^1(\Delta, \mathcal{O}^*) = 0. \tag{8.36}$$

Corollary 8.87 *Each multiplicative holomorphic Cousin Problem on a polydisk can be solved after passing to a smaller covering.*

Proof Let $\mathcal{U}$ be a covering of a polydisk Δ, $h = (h_{ij})$ be a holomorphic multiplicative cocycle with respect to the covering $\mathcal{U}$. It represents a cohomology class $[h]_{\mathcal{U}} \in H^1_{\mathcal{U}}(\Delta, \mathcal{O}^*)$. Its projection $[h] \in H^1(\Delta, \mathcal{O}^*)$ vanishes, by (8.36). Therefore, there exists a smaller covering $\mathcal{V} < \mathcal{U}$ such that the restriction $\rho_\phi h$ of the cocycle h to the intersections of the elements of the covering $\mathcal{V}$ is a coboundary: it represents zero cohomology class in $H^1_{\mathcal{V}}(\Delta, \mathcal{O}^*)$. Hence, the corresponding Multiplicative Cousin problem can be solved for the covering $\mathcal{V}$, see Example 8.69. The corollary is proved. □

Corollary 8.88 *Every holomorphic hypersurface A in a polydisk Δ has a global defining holomorphic function: there exists a holomorphic function $f : \Delta \to \mathbb{C}$ such that $A = \{f = 0\}$ and for every $x \in A$ the function f generates the ideal $I_A(x)$.*

Proof Let $\mathcal{U}$ be a covering of the polydisk Δ by open sets U_j with holomorphic functions $f_j : U_j \to \mathbb{C}$ such that for every j one has $A \cap U_j = \{f_j = 0\}$ and f_j generates the ideal $I_A(x)$ for every $x \in A \cap U_j$: these U_j and f_j were constructed just after Corollary 8.41. Set $h_{ij} = \frac{f_j}{f_i}$ whenever $U_i \cap U_j \neq \emptyset$. Let $\mathcal{V} < \mathcal{U}$ be a smaller covering by open sets V_α for which the corresponding restriction of the cocycle h is trivial in the cohomology $H^1_{\mathcal{V}}(\Delta, \mathcal{O}^*)$. The restricted cocycle is given by the same formula

$$(\rho_\phi h)_{\alpha\beta} = \frac{f_{\phi(\beta)}}{f_{\phi(\alpha)}}.$$

Let (g_α) be its trivialization:

$$g_\alpha : V_\alpha \to \mathbb{C}^* \text{ are holomorphic functions, } \frac{g_\beta}{g_\alpha} = h_{\alpha\beta} \text{ on } V_\alpha \cap V_\beta.$$

Then for every two intersected sets V_α and V_β one has

$$\frac{f_{\phi(\beta)}}{g_\beta} = \frac{f_{\phi(\alpha)}}{g_\alpha} \text{ on } V_\alpha \cap V_\beta.$$

Thus, the latter ratios are restrictions to V_α of a global holomorphic function $f : \Delta \to \mathbb{C}$, which has the required properties by construction. The corollary is proved. □

Corollary 8.89 *Each holomorphic function on a hypersurface A in a polydisk Δ is the restriction to A of a global holomorphic function on Δ.*

Proof We have proved the statement of the corollary under the condition of the existence of global defining function (Theorem 8.54). But we have just proved that a global defining function always exists. This proves the corollary. $\square$

Let M be a complex manifold, $\mathcal{E}^{p,q}$ be the sheaf of (germs of) C^∞-smooth (p,q)-forms on M, $\mathcal{E}^0$ be the sheaf of C^∞-smooth functions. The sheaf $\bar\partial\mathcal{E}^{p,q-1}$ is the subsheaf in $\mathcal{E}^{p,q}$ that is the image of the sheaf $\mathcal{E}^{p,q-1}$ under the antiholomorphic differential $\bar\partial$ (which is a sheaf homomorphism). This is the sheaf of locally $\bar\partial$-exact (p,q)-forms. Equivalently, this is the sheaf of $\bar\partial$-closed (p,q)-forms, since the notions of $\bar\partial$-exactness and $\bar\partial$-closedness coincide locally: each point of the manifold has a neighborhood U biholomorphically equivalent to a polydisk; each $\bar\partial$-closed (p,q)-form on U is $\bar\partial$-exact by the $\bar\partial$-Poincaré Lemma. Let Ω^p denote the sheaf of holomorphic $(p,0)$-forms, $\Omega^0=\mathcal{O}$. One has the following exact sequences:

$$0\to\mathcal{O}\to\mathcal{E}^0\to\bar\partial\mathcal{E}^0\to 0, \tag{8.37}$$

$$0\to\Omega^p\to\mathcal{E}^{p,0}\to\bar\partial\mathcal{E}^{p,0}\to 0 \tag{8.38}$$

$$0\to\bar\partial\mathcal{E}^{p,q-1}\to\mathcal{E}^{p,q}\to\bar\partial\mathcal{E}^{p,q}\to 0. \tag{8.39}$$

The left homomorphisms are the natural inclusions. The right ones are the $\bar\partial$-operators.

Theorem 8.90 *For every complex manifold M one has*

$$H^k(M,\mathcal{E}^{p,q})=0 \text{ for all } k\geq 1,\ p,\ q, \tag{8.40}$$

$$H^{0,1}(M)=H^1(M,\mathcal{O}),\ H^{p,q}(M)=H^q(M,\Omega^p) \text{ for every } q\geq 1,\ p\geq 0. \tag{8.41}$$

Proof Statement (8.40) is proved analogously to the proof of Theorem 8.48 (cf. Task 5, part 2, Problem 3). The first equality in (8.41) is a particular case of the second one; thus it suffices to prove the second one. The long exact cohomology sequence corresponding to (8.38) together with vanishing of the middle cohomology $H^k(M,\mathcal{E}^{p,0})$ for $k\geq 1$ yields

$$0\to H^k(M,\bar\partial\mathcal{E}^{p,0})\to H^{k+1}(M,\Omega^p)\to 0.$$

Therefore,

$$H^k(M, \bar{\partial}\mathcal{E}^{p,0}) = H^{k+1}(M, \Omega^p) \text{ for every } k \geq 1.$$

Similarly, the exact cohomology sequence corresponding to (8.39) yields

$$H^k(M, \bar{\partial}\mathcal{E}^{p,q}) = H^{k+1}(M, \bar{\partial}\mathcal{E}^{p,q-1}) \text{ for every } k, q \geq 1.$$

The two last statements together imply that

$$H^q(M, \Omega^p) = H^1(M, \bar{\partial}\mathcal{E}^{p,q-2}) \text{ for every } q \geq 2. \tag{8.42}$$

On the other hand, the initial part of the cohomology sequence associated to (8.39) yields

$$H^0(M, \mathcal{E}^{p,q-1}) \to H^0(M, \bar{\partial}\mathcal{E}^{p,q-1}) \to H^1(M, \bar{\partial}\mathcal{E}^{p,q-2}) \to 0.$$

Therefore, the third term of the latter sequence is isomorphic to the quotient of the middle term over the image of the first term. This is exactly the quotient of the space of global $\bar{\partial}$-closed (p,q)-forms over the space of global $\bar{\partial}$-exact (p,q)-forms, the Dolbeault cohomology group $H^{p,q}(M)$:

$$H^{p,q}(M) = H^1(M, \bar{\partial}\mathcal{E}^{p,q-2}).$$

Substituting the latter equality to (8.42) yields (8.41). The theorem is proved. $\square$

Corollary 8.91 *One has*

$$H^q(M, \Omega^p) = 0 \textit{ whenever } H^{p,q}(M) = 0. \tag{8.43}$$

Remark 8.92 Cousin Theorem 8.50 and its proof imply that if $H^{0,1}(M) = 0$, then every holomorphic Additive Cousin Problem can be solved. In fact, the above proof of Theorem 8.90 is just a sheaf-theoretic translation of the proof of Theorem 8.50.

Remark 8.93 Applying the arguments from the proof of Theorem 8.90 to a real manifold and the usual d-differential $d : \mathcal{E}^k \to \mathcal{E}^{k+1}$ acting on the space of differential k-forms and to the short exact sequence

$$0 \to d\mathcal{E}^{k-1} \to \mathcal{E}^k \to d\mathcal{E}^k \to 0$$

yields a short proof of de Rham Theorem. See Task 5, part 2, Problem 2.

8.8 Coherent analytic sheaves, Stein manifolds and Cartan Theorems A, B

Let M be a complex manifold. In the present subsection we will deal with the sheaves of $\mathcal{O}$-modules over M.

Definition 8.94 A sheaf of $\mathcal{O}$-modules is a sheaf $\mathcal{J}$ of Abelian groups over M such that each fiber $\mathcal{J}_x = \pi^{-1}(x)$ is equipped with the multiplication by elements of the sheaf $\mathcal{O}$, germs of holomorphic functions at x, so that the multiplication is continuous on the product of $\mathcal{J} \times \mathcal{O}$ of the sheaf spaces. A sheaf $\mathcal{J}$ of $\mathcal{O}$-modules is called *coherent,* if every $x \in M$ has a neighborhood $U = U(x) \subset M$ with the following properties:

1) the sheaf $\mathcal{J}|_U$ is finitely generated over $\mathcal{O}$ by sections: there exist an $m = m(U) \in \mathbb{N}$ and an epimorphic homomorphism $g : \mathcal{O}^m|_U \to \mathcal{J}|_U$;

2) the kernel $\operatorname{Ker} g$ is finitely generated: there exist a $k = k(U, m) \in \mathbb{N}$ and an epimorphic homomorphism $f : \mathcal{O}^k|_U \to \operatorname{Ker} g|_U$: thus we have the exact sequence

$$\mathcal{O}^k|_U \to \mathcal{O}^m|_U \to \mathcal{J}|_U \to 0.$$

Example 8.95 The following are sheaves of $\mathcal{O}$-modules:

- the sheaf $\mathcal{O}^m$ of m-dimensional holomorphic vector functions;
- the sheaf of holomorphic sections of a holomorphic vector bundle (which is *locally free:* locally isomorphic to the sheaf $\mathcal{O}^m$, where m is the dimension of the bundle);
- the sheaf $\mathcal{J}(A)$ of holomorphic functions on M vanishing on an analytic subset $A \subset M$;
- the sheaf $\mathcal{O}_A$ of holomorphic functions on an analytic set A, see Definition 8.53. We extend it as a sheaf over the base M by putting $\mathcal{O}_A|_{M\setminus A} = 0$.

All of them are coherent analytic sheaves. Indeed, there is an obvious exact sequence

$$0 \to \mathcal{O}^m \to \mathcal{O}^m \to 0$$

with identity isomorphism in the middle. Hence, the sheaf $\mathcal{O}^m$ is coherent. The sheaf $\mathcal{J}(A)$ is finitely generated, as a local ideal $I_A(x)$ of every germ of analytic set. That is, each $x \in M$ has a neighborhood U for which there exist $m \in \mathbb{N}$ and an epimorphic homomorphism $g : \mathcal{O}^m \to \mathcal{J}$. Let $g_1, \dots, g_m \in \Gamma(\mathcal{J}|_U)$ denote the images of the basic constant unit vector functions $(0 \dots 010 \dots)$. One has

$$\operatorname{Ker} g = \{(h_1, \dots, h_n) \mid h_j \text{ are holomorphic and } \sum_j h_j g_j = 0.$$

It appears that the latter kernel is also finitely generated, while restricted to a small neighborhood of the point x.

Remark 8.96 It is known that if

$$0 \to A \to B \to C \to 0$$

is a short exact sequence of sheaves of $\mathcal{O}$-modules and at least two of the sheaves A, B, C are coherent, then the third sheaf is also coherent.

Theorem 8.97 (Cartan's Theorem A). *For every coherent analytic sheaf $\mathcal{J}$ on a Stein manifold M and every $x \in M$ the space of sections $H^0(M, \mathcal{J})$ generates $\mathcal{J}_x = \pi^{-1}(x)$ as an $\mathcal{O}$-module.*

Theorem 8.98 (Cartan's Theorem B). *For every coherent analytic sheaf $\mathcal{J}$ on a Stein manifold M one has $H^q(M, \mathcal{J}) = 0$ for all $q \geq 1$.*

Corollary 8.99 *For every Stein manifold M one has*

$$H^q(M, \mathcal{O}) = 0, \ H^q(M, \mathcal{O}^*) = H^{q+1}(M, \mathbb{Z}) \text{ whenever } q \geq 1.$$

In particular, $H^1(M, \mathcal{O}^*) = 0$, whenever $H^2(M, \mathbb{Z}) = 0$.

Proof The first equality follows from Cartan's Theorem B and coherence of the sheaf $\mathcal{O}$. The second equality follows from the first one and the exact cohomological sequence associated to the short sequence (8.34). □

Theorem 8.100 *Let M be a Stein manifold, $A \subset M$ be an analytic subset. Then there exists a (may be infinite) family of global holomorphic functions $f_i : M \to \mathbb{C}$, $i \in I$, such that*

$$A = \{f_i = 0 \mid i \in I\}.$$

Every holomorphic function on A is the restriction to A of a global holomorphic function on M.

Proof The first statement of the theorem is equivalent to the statement saying that for every $x \in M$ there exists a holomorphic function $f : M \to \mathbb{C}$ vanishing on A that does not vanish at x. To prove this, consider the sheaf $\mathcal{J}(A)$ of functions vanishing on A. For every $x \notin A$ the $\mathcal{O}$-module $\mathcal{J}_x$ is generated by global sections $M \to \mathcal{J}(A)$: global holomorphic functions on M vanishing on A. In particular, the germ at x of constant function $f \equiv 1$

is a finite linear combination $\sum_j g_j f_j$, where $f_j : M \to \mathbb{C}$ are holomorphic functions vanishing on A, g_j are germs at x of holomorphic functions. In particular, $f_j(x) \neq 0$ for some j. This proves the first statement of the theorem.

Let us prove the second statement of the theorem. One has the short exact sequence

$$0 \to \mathcal{J}(A) \to \mathcal{O} \to \mathcal{O}_A \to 0.$$

The corresponding long exact cohomological sequence contains the part

$$H^0(M, \mathcal{O}) \to H^0(M, \mathcal{O}_A) \to H^1(M, \mathcal{J}(A)).$$

The left mapping is given by the restriction of the functions to A. The sheaf $\mathcal{J}(A)$ being coherent, one has $H^1(M, \mathcal{J}(A)) = 0$, by Cartan's Theorem B. This implies that the restriction mapping $H^0(M, \mathcal{O}) \to H^0(M, \mathcal{O}_A)$ is surjective, which is exactly the second statement of the theorem. $\square$

Theorem 8.101 *Let M be a Stein manifold, and let $U \subset M$ be an $H(U)$-convex domain. Then every holomorphic function $f : U \to \mathbb{C}$ is a limit of a sequence of holomorphic functions $f_k : M \to \mathbb{C}$ converging to f uniformly on compact subsets in U.*