Topological Vector Spaces

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The traditional functional analysis deals mostly with Banach spaces and, in particular, Hilbert spaces. However, many classical vector spaces have canonical topologies that cannot be determined by a single norm. For example, many spaces of smooth functions, holomorphic functions, and distributions belong to the above class. Such spaces are the subject of the theory of topological vector spaces. Although the golden age of topological vector spaces was in the 1950ies, their theory is still evolving nowadays, contrary to a stereotyped view coming from incompetent sources. The current development of topological vector spaces is directed not so much towards general theory as towards applications in PDEs and in complex analytic geometry.

A closely related field is the theory of bornological spaces. Roughly, a bornological space is a vector space in which bounded subsets are defined axiomatically. Each topological vector space carries several natural bornologies, and conversely, each bornological space has several natural topologies. Bornological spaces appeared in the 1960ies, but have been nearly forgotten soon. However, quite recently (about 10 years ago) bornological spaces have proved to be very useful in some problems of noncommutative geometry, specifically in bivariant K-theory and in cyclic homology theory. This resulted in a considerable surge of interest in bornological spaces and led to a revival and development of their theory.

We plan to discuss the basics of the theory of topological and bornological vector spaces, including some applications. The applications will be chosen by agreement with the audience.

Prerequisites. Basic functional analysis (Banach and Hilbert spaces, bounded linear operators).

Syllabus. Topological vector spaces. Seminorms and locally convex spaces. Basic constructions. Examples: spaces of smooth functions, holomorphic functions, and distributions. Metrizability, completeness. Fréchet spaces. Duality. Weak topology, Mackey topology, strong topology. Topological tensor products. Nuclear operators. Nuclear spaces. Bornological spaces. Examples, constructions, and comparison of the bornological and topological categories. Applications (for the audience to choose from): to distribution theory (the Schwartz kernel theorem), to complex analytic geometry (the Cartan-Serre finiteness theorem on the cohomology of coherent analytic sheaves on a compact manifold), to cyclic homology (entire and analytic cyclic homology for bornological algebras).