Goresky-MacPherson cohomology

This is a course for seniors and graduate students. The prerequisites include homological algebra (derived categories), arithmetics (Galois theory), and algebraic geometry.

Singular cohomology of a complex variety is the cohomology with coefficients in the constant constructible sheaf. If a variety is singular, it carries a canonical constructible complex of sheaves (Goresky-MacPherson sheaf) whose hypercohomology is in many respects better than the usual singular cohomology. For instance, the Poincaré duality holds for this hypercohomology. If we consider the étale cohomology of projective varieties over a finite field $\mathbb{F}_q$ then the eigenvalues of the Frobenius automorphism of the degree n Goresky-MacPherson cohomology have absolute value $q^{n/2}$ (Riemann-Weil conjecture).

Apart from the Goresky-MacPherson sheaf there are other constructible complexes with similar properties; they form an abelian category of perverse sheaves. This category possesses many favorable functorial properties with respect to morphisms of the underlying algebraic varieties. For example, the direct image of a semismall morphism preserves perversity, and the direct image of a semisimple perverse sheaf with respect to a proper morphism is again semisimple (a corollary of the Riemann-Weil conjecture). The vanishing and nearby cycles preserve perversity. Iterating the vanishing and nearby cycles, one can describe the whole category of perverse sheaves in terms of linear algebra (Beilinson's gluing theorem). For instance, the category of quantum group's representations is equivalent to the category of perverse (factorizable) sheaves on the configuration space of a Riemann surface. The functoriality of Goresky-MacPherson extenstion immediately gives a construction of Springer representation of the Weyl group in the cohomology of the fixed point set of a nilpotent vector field on the flag variety of a semisimple Lie group. The Fourier-Deligne transform preserves perversity (and constitutes one of the key steps in the proof of the Riemann-Weil conjecture). Perverse sheaves on the Schubert varieties in the affine Grassmannian of a semisimple Lie group are closed with respect to convolution and form a tensor category equivalent to the category of representations of the Langlands dual group.

All in all, the notion of perverse sheaves is one of the greatest mathematical discoveries of the last quarter of the 20-th century. The big part of the successes of the representation theory (classification and calculation of irreducible characters of complex and modular representations of finite and p-adic Lie groups, quantum groups, and affine Lie algebras; construction of the automorphic representations of adelic groups and other advances of the geometric representation theory) is due to the applications of perverse sheaves.