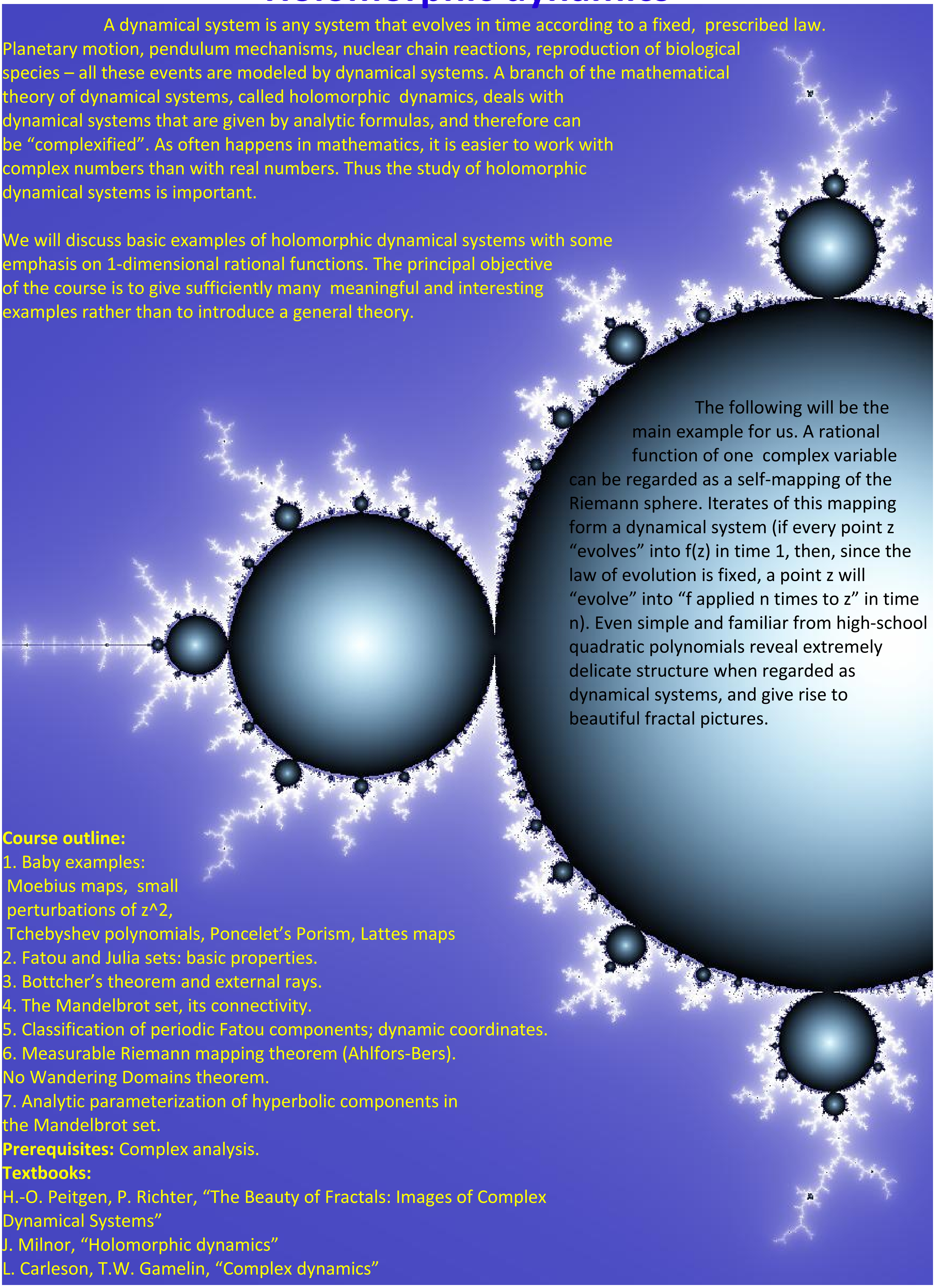
Holomorphic dynamics



A dynamical system is any system that evolves in time according to a fixed, prescribed law. Planetary motion, pendulum mechanisms, nuclear chain reactions, reproduction of biological species – all these events are modeled by dynamical systems. A branch of the mathematical theory of dynamical systems, called holomorphic dynamics, deals with dynamical systems that are given by analytic formulas, and therefore can be “complexified”. As often happens in mathematics, it is easier to work with complex numbers than with real numbers. Thus the study of holomorphic dynamical systems is important.

We will discuss basic examples of holomorphic dynamical systems with some emphasis on 1-dimensional rational functions. The principal objective of the course is to give sufficiently many meaningful and interesting examples rather than to introduce a general theory.

The following will be the main example for us. A rational function of one complex variable can be regarded as a self-mapping of the Riemann sphere. Iterates of this mapping form a dynamical system (if every point z “evolves” into f(z) in time 1, then, since the law of evolution is fixed, a point z will “evolve” into “f applied n times to z” in time n). Even simple and familiar from high-school quadratic polynomials reveal extremely delicate structure when regarded as dynamical systems, and give rise to beautiful fractal pictures.

Course outline:
1. Baby examples:
 Moebius maps, small perturbations of z^2,
 Tchebyshev polynomials, Poncelet’s Porism, Lattes maps
2. Fatou and Julia sets: basic properties.
3. Bottcher’s theorem and external rays.
4. The Mandelbrot set, its connectivity.
5. Classification of periodic Fatou components; dynamic coordinates.
6. Measurable Riemann mapping theorem (Ahlfors-Bers).
No Wandering Domains theorem.
7. Analytic parameterization of hyperbolic components in the Mandelbrot set.

Prerequisites: Complex analysis.

Textbooks:

H.-O. Peitgen, P. Richter, “The Beauty of Fractals: Images of Complex Dynamical Systems”
J. Milnor, “Holomorphic dynamics”
L. Carleson, T.W. Gamelin, “Complex dynamics”