Functional Analysis 1

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Functional analysis studies infinite-dimensional vector spaces endowed with a norm (or, more generally, with a topology), operators between such spaces, and representations of algebraic structures on such spaces. It comprises many various fields, both classical and new. Among the classical areas are the spectral theory of linear operators, the general theory of Banach and topological vector spaces, distribution theory, operator algebra theory, ergodic theory... Among relatively new and rapidly developing fields, we would like to mention noncommutative geometry à la Connes, operator space theory (which is sometimes called "quantum functional analysis"), and the theory of locally compact quantum groups. Functional analysis has numerous applications in differential equations, harmonic analysis, representation theory, geometry, topology, calculus of variations, optimization, quantum physics, etc.

Syllabus

1. **Normed and Banach spaces.** Normed spaces. Bounded linear operators. Basic examples. Constructions of normed spaces. Banach spaces. The completeness of classical spaces. Constructions preserving completeness. Completion.

2. **Hilbert spaces.** Projections and orthogonal complements. Orthonormal systems. Bessel's Inequality. Orthonormal bases. Parseval's Identity. The Riesz-Fischer Theorem. Classification of Hilbert spaces.

3. **Linear functionals.** The Hahn-Banach Theorem. Separation of convex sets. Dual spaces and adjoint operators. The duals of classical Banach spaces. The canonical embedding into the bidual. Reflexivity.

4. **Banach's theorems and duality.** The Open Mapping and the Closed Graph Theorems. The Uniform Boundedness Principle (the Banach-Steinhaus Theorem). Complemented subspaces. Annihilators, preannihilators. The duals of subspaces and quotients. Relations between properties of operators and their adjoints.

5. **Elementary spectral theory.** The spectrum of an algebra element. Banach algebras. The nonemptiness and the compactness of the spectrum. The Gelfand-Mazur Theorem. Spectral radius. The point spectrum, the continuous spectrum, and the residual spectrum of a linear operator. Spectra and duality. Calculations for classical operators.