Harmonic analysis and unitary representations

(Spring 2014)

A. Yu. Pirkovskii

Harmonic analysis on groups and unitary representation theory are closely related areas of mathematics, complementary to each other. They play an important role in analysis, geometry, topology, physics, and other fields of science. In essence, they grew out of two classical topics that are usually studied by undergraduate students in mathematics. The two topics are the theory of trigonometric Fourier series and the representation theory (over $\mathbb{C}$) of finite groups. Among other things, we plan to explain what the above topics have in common, what the representation theory of compact groups looks like, what the Tannaka-Krein duality is, and what all this has to do with the Fourier transform. We are also going to construct harmonic analysis on locally compact abelian groups. This theory includes the Pontryagin duality and generalizes the Fourier transform theory on the real line. As an auxiliary material, the basics of Banach algebra theory will also be given. In conclusion, we intend to discuss (mostly by examples) some aspects of harmonic analysis and unitary representation theory for noncompact, nonabelian groups.

Prerequisites. The Lebesgue integration theory and the basics of functional analysis. Some knowledge of the representation theory of finite groups will also be helpful.

Syllabus

1. **Introduction.** A toy example: harmonic analysis on a finite abelian group. Classical examples: harmonic analysis on the integers, on the circle, and on the real line.

2. **The main objects.** Topological groups. The Haar measure. A relation between the left and right Haar measures. Unitary representations. The general Fourier transform.

3. **Banach algebras.** The L^1-algebra of a locally compact group. The spectrum of a Banach algebra element. Commutative Banach algebras, the Gelfand spectrum, the Gelfand transform. Basics of C^*-algebra theory. The C^*-algebra of a locally compact group. The 1st Gelfand-Naimark theorem.

4. **Locally compact abelian groups.** The dual group. The Fourier transform as a special case of the Gelfand transform. The Plancherel theorem. The Pontryagin duality. An application: the Poisson summation formula.

5. **Compact groups.** The averaging procedure. Irreducible representations are finite-dimensional. Decomposing unitary representations into irreducibles. The Peter-Weyl theorem. The orthogonality relations. The Fourier transform and its inverse. The Plancherel theorem. The Tannaka-Krein duality.

6. **Locally compact nonabelian groups.** The Heisenberg group and its irreducible representations. The Stone-von Neumann theorem. A relation to coadjoint orbits. The Fourier transform and the Plancherel theorem for the Heisenberg group. A survey of $\mathrm{SL}_2(\mathbb{R})$: structure, irreducible representations, the Plancherel theorem.