An introduction to generalised cohomology theories and applications

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Generalised (co)homology theories are invariants of topological spaces. They are similar to the usual (singular) homology theory in that they satisfy all of the Eilenberg-MacLane axioms apart from the dimension axion, which states that the (co)homology of a point is zero in all non-zero degrees. It turns out that such invariants do in fact exist. Moreover, they are computable (in some cases) and meaningful, meaning that they know something about topological spaces that the ordinary cohomology theory doesn't. In the first part of the course (semester 1) we will look at some of the basic properties of generalised (co)homology theories with a focus on the (topological) K-theory and its applications such as Hopf invariant 1 problem and vector fields on spheres. The prerequisites for this part are 1. the material covered in the second year topology course at HSE (or almost any other introductory topology course), and 2. some familiarity with the ordinary (co)homology.

In the second part (semester 2) we intend to cover the Atiyah-Singer index theorem and its applications. The prerequisites are some differential geometry (connections, Riemannian metrics, curvature) and functional analysis (basically, the material covered in the "Functional analysis" course, part 1). As an alternative to that we could instead cover cohomology operations, the Adams spectral sequence and its applications.

Here is a tentative syllabus.

Part 1

- Vector bundles and characteristic classes (summary of main results).
- The definition and first properties of K-groups.
- The complex Bott theorem.
- The Adams operations and non-existence of maps with Hopf invariant 1. Applications to division algebras.
- The real Bott theorem and applications to vector fields on spheres.
- Generalised (co)homology theories. Ω-spectra and Brown's representability theorem.
- Stable homotopy groups as a generalised cohomology theory. Realisability of cohomology classes by submanifolds.
- The Atiyah-Hirzebruch spectral sequence.

Part 2

- Elliptic operators. Spinor and Clifford bundles. The Dirac operator.
- Classical elliptic operators (the Atiyah-Singer operator, the Hodge Laplacian, the signature operator).
- Sobolev theorems.
- Pseudodifferential operators.
- Fundamental results for elliptic operators. The Hodge decomposition theorem.
- The topological invariance of the index.
- Generalisations of the notion of index.
- Proofs of the index theorems.
- Applications of the index theorems: integrality theorems and obstructions to the existence of immersions and embeddings.

Here are the main references

- *Lectures on K-theory* by M. Atiyah.
- *A coupse in homotopy theory* by D.B. Fuchs and A.T.Fomenko.
- *Algebraic Topology* by A. Hatcher. `http://www.math.cornell.edu/~hatcher/AT/ATpage.html`.
- *Vector bundles and K-theory* by A. Hatcher. `http://www.math.cornell.edu/~hatcher/VBKT/VBpage.html`.
- *Spin geometry* by H.B. Lawson and M.-L. Michelsohn.
- *Characteristic classes* by J. Milnor and J. Stasheff.
- *Vector bundles and applications* by A.S.Mischenko.