FUNCTIONAL ANALYSIS

1.1. *Let* $p, q \in (1, +\infty)$, *and* $\frac{1}{p} + \frac{1}{q} = 1$.

1) Prove the Young's inequality:

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \qquad (a, b \geq 0).$$

2) Deduce from the Young's inequality the Hölder's inequality:

$$\sum_{i=1}^{n} |x_i y_i| \leq \|x\|_p \|y\|_q \qquad (x, y \in \mathbb{K}^n).$$

3) Deduce from the Hölder's inequality the Minkowski's inequality:

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p \qquad (x, y \in \mathbb{K}^n).$$

1.2. *Draw the unit disc on the plane* $\mathbb{R}^2$ *endowed with the norm* $\| \cdot \|_p$ *for varying* $p \in [1, +\infty]$. *Take special attention to the cases* $p = 1$, $p = 2$, $p = \infty$. *What happens with unit disc when* p *grows?*

1.3. *Let* $1 \leq p \leq q \leq +\infty$.

1) Show that $\| \cdot \|_q \leq \| \cdot \|_p$ *on* $\mathbb{K}^n$.

2) Prove that there exists a constant $C = C_{n,p,q} > 0$ *such that* $\| \cdot \|_p \leq C \cdot \| \cdot \|_q$ *on* $\mathbb{K}^n$.

3) Can one choose this constant to be independent of n*?*

4) Find a minimal constant $C_{n,p,q}$ *with this property. Interpret the answer as norm of an operator.*

1.4. *Let* c_{00} *be the space of finite sequences, i.e., such sequences* $x = (x_n)$ *that* $x_n = 0$ *for all sufficiently big* n. *Are the norms* $\| \cdot \|_p$ *and* $\| \cdot \|_q$ *on* c_{00} *equivalent on* c_{00} *for* $p \neq q$*?*

1.5. *Show that a sequence* $(x^{(k)})$ *in the space* $\mathbb{K}^n$ *converges to a vector* $x \in \mathbb{K}^n$ *in the norm* $\| \cdot \|_p$ *(where* $1 \leq p \leq +\infty$*) if and only if it converges to* x *componentwise.*

1.6. *Show that* c_0 *is closed in* ℓ^∞. *What is the closure of* ℓ^p *in* ℓ^∞*?*

1.7. *Let* $1 \leq p \leq q \leq +\infty$. *Prove that* $\ell^p \subset \ell^q$, *but* $\ell^p \neq \ell^q$ *if* $p \neq q$. *What is the norm of the inclusion operator* $\ell^p \hookrightarrow \ell^q$*?*

1.8. *Prove that a sequence* (f_n) *in the space* $\ell^\infty(S)$ *converges to* $f \in \ell^\infty(S)$ *if and only if it converges to* f *uniformly.*

1.9. *Prove that* $C_0(\mathbb{N}) = c_0$.

1.10. *Let* X *be a seminormed space and let* $N = \{x \in X \ : \ \|x\| = 0\}$. *Show that (i)* N *is a subspace of* X*; (ii) the formula*

$$\|x + N\|^\wedge = \|x\| \qquad (x \in X)$$

gives a well-defined norm on X/N.

1.11. *Let* (X, μ) *be a measurable space and let* $p, q \in (1, +\infty)$ *be such that* $\frac{1}{p} + \frac{1}{q} = 1$.

1) Prove that if $f \in \mathcal{L}^p(X, \mu)$ *and* $g \in \mathcal{L}^q(X, \mu)$ *then* fg *is integrable and the Hölder's inequality holds:*

$$\int_X |fg| \leq \|f\|_p \|g\|_q.$$

2) Deduce from the Hölder's inequality that (i) $\mathcal{L}^p(X, \mu)$ *is a vector space, (ii) the Minkowski's inequality holds:*

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p \qquad (x, y \in \mathbb{K}^n) \qquad (f, g \in \mathcal{L}^p(X, \mu)).$$

1.12. *Let* $1 \leq p \leq q \leq +\infty$.

1) Prove that there exists a constant $C = C_{a,b,p,q} > 0$ *such that* $\| \cdot \|_p \leq C \cdot \| \cdot \|_q$ *on the space* $C[a, b]$.

2) Find a minimal constant $C_{a,b,p,q}$ *with this property. Interpret the answer as norm of an operator.*

3) Are the norms $\| \cdot \|_p$ *and* $\| \cdot \|_q$ *on the space* $C[a, b]$ *equivalent for* $p \neq q$*?*

1.13. *Check that a measurable function is essentially bounded if and only if it is equivalent to a bounded measurable function.*

1.14. *Let* (X, μ) *be a measurable space and let* f *be a non-negative essentially bounded function on* X. *Recall that essential upper bound is defined by*

$$\text{ess } \sup f = \inf\{\sup_{x \in E} f(x) \ : \ E \subset X, \ \mu(X \setminus E) = 0\}.$$

Prove that inf *in this formula is attained. Conclude that* ess $\sup f = 0$ *if and only if* $f = 0$ *almost everywhere.*

1.15. *Prove that if* $f = g$ *almost everywhere then* ess $\sup f =$ ess $\sup g$.

1.16. *Let* $f \in C[a, b]$. *Prove that* ess $\sup |f| = \sup_{x \in [a,b]} |f(x)|$.

1.17. *Prove that* $\mathcal{L}^\infty(X, \mu)$ *is a vector space and* $\|f\| =$ ess $\sup |f|$ *is a seminorm on* $\mathcal{L}^\infty(X, \mu)$.

1.18. *Let* $\mu(X) < \infty$. *Prove that* $\mathcal{L}^q(X, \mu) \subseteq \mathcal{L}^p(X, \mu)$ *for any* $1 \leq p \leq q \leq +\infty$. *What is the norm of the inclusion operator* $\mathcal{L}^q(X, \mu) \hookrightarrow \mathcal{L}^q(X, \mu)$*?*

1.19. *Prove that* $\mathcal{L}^p[a, b] \neq \mathcal{L}^q[a, b]$ *for any* $p \neq q$.

1.20. *Let* $X = \mathbb{N}$ *and* μ *be the "counting" measure on the* σ*-algebra of all subsets of* $X = \mathbb{N}$, *given by* $\mu(A) = |A|$. *Verify that* $L^p(X, \mu) = \ell^p$ *for all* $1 \leq p \leq q \leq +\infty$. *Compare this with the result of Problem 1.7 and conclude that the result of Problem 1.18 is not valid if* $\mu(X) = \infty$.

1.21. *Show that* $\mathcal{L}^p(\mathbb{R}) \not\subseteq \mathcal{L}^q(\mathbb{R})$ *for any* $p \neq q$. *It is instructive to compare this result with Problems 1.7 and 1.18.*