2.1. Prove that the following *parallelogram rule*

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

holds in every inner product space.

2.2. Show that the norm on the spaces $(\mathbb{C}^n, \|\cdot\|_p)$, ℓ^p, $(C[a,b], \|\cdot\|_p)$, $L^p(X,\mu)$ (where (X,μ) is a measure space containing at least two disjoint measurable sets of positive measure) is not generated by an inner product (unless $n=1$, $p=2$).

2.3. Generalize the parallelogram rule to n vectors.

2.4. Show that the norm on the spaces ℓ^p, $(C[a,b], \|\cdot\|_p)$, $L^p(X,\mu)$ (where (X,μ) is a measure space containing infinitely many disjoint measurable sets of positive measure) is not equivalent to a norm generated by an inner product (unless $p=2$).

2.5. Consider the vector space $H = C[-1,1]$ with the inner product $\langle f,g\rangle = \int_{-1}^{1} f(t)\overline{g(t)}\,dt$. Let

$$H_0 = \left\{ f \in H : \int_{-1}^{0} f(t)\,dt = \int_{0}^{1} f(t)\,dt \right\}.$$

(a) Prove that H_0 is a closed vector subspace of H.
(b) Does the equality $H = H_0 \oplus H_0^\perp$ hold?

2.6. Prove that every incomplete inner product space H has a closed vector subspace H_0 such that $H_0 \oplus H_0^\perp \neq H$.

2.7. Let $C_c^\infty(a,b)$ be the space of smooth compactly supported functions on the interval (a,b). Prove that for each $p \in [1,\infty)$ $C_c^\infty(a,b)$ is dense in $L^p[a,b]$.

Definition 2.1. Let $f \in L^2[a,b]$. A function $f' \in L^2[a,b]$ is a *weak derivative* of f if

$$\int_a^b f'\varphi\,dt = -\int_a^b f\varphi'dt$$

for all $\varphi \in C_c^\infty(a,b)$.

2.8. Prove that if $f \in L^2[a,b]$ has a weak derivative f', then f' is unique (as an element of $L^2[a,b]$).

2.9. The *Sobolev space* $W^{1,2}(a,b)$ consists of all $f \in L^2[a,b]$ that have a weak derivative $f' \in L^2[a,b]$. Prove that $W^{1,2}(a,b)$ is a Hilbert space with respect to the inner product

$$\langle f,g\rangle = \int_a^b (f\bar{g} + f'\bar{g}')\,dt.$$

2.10. Let (e_i) be an orthonormal family in a Hilbert space H. Prove that (e_i) is an orthonormal basis if and only if for each $x \in H$ the Parseval identity $\|x\|^2 = \sum_i |\langle x, e_i\rangle|^2$ holds.

2.11. Choose $t_0 \in [a,b]$, and consider the linear functional

$$F\colon (C[a,b], \|\cdot\|_p) \to \mathbb{K}, \quad F(x) = x(t_0).$$

(a) Find all $p \in [1,+\infty]$ such that F is bounded. **(b)** Find $\|F\|$.

2.12. Let X be either $L^p[0,1]$ $(1 \leqslant p < +\infty)$ or $C[0,1]$. Define $T\colon X \to X$ by

$$(Tf)(x) = \int_0^x f(t)\,dt \qquad (f \in X).$$

(a) Prove that T is bounded. **(b)** Find $\|T\|$ in the cases where $X = C[0,1]$ and $X = L^1[0,1]$.

Remark. If the above operator T acts on $L^2[0,1]$, then $\|T\| = 2/\pi$. We will be able to prove this in due course.

2.13. Let $I = [a,b]$, and let $K \in C(I \times I)$. The *integral operator* $T\colon C(I) \to C(I)$ is given by

$$(Tf)(x) = \int_a^b K(x,y) f(y)\,dy.$$

Prove that T takes $C(I)$ to $C(I)$, that T is bounded, and that $\|T\| \leqslant \|K\|_\infty (b-a)$.

2.14. Let (X,μ) be a measure space, and let $K \in L^2(X \times X, \mu \times \mu)$. The *Hilbert-Schmidt integral operator* $T\colon L^2(X,\mu) \to L^2(X,\mu)$ is given by

$$(Tf)(x) = \int_X K(x,y) f(y)\,d\mu(y).$$

Prove that T takes $L^2(X,\mu)$ to $L^2(X,\mu)$, that T is bounded, and that $\|T\| \leqslant \|K\|_2$.

2.15. Define a linear functional F on $(C[0,1], \|\cdot\|_\infty)$ by

$$F(f) = 2f(0) - 3f(1) + \int_0^1 f(t)\,dt.$$

(a) Prove that F is bounded. **(b)** Find $\|F\|$.

2.16. Let X, Y be normed spaces. Suppose that X is finite-dimensional. Prove that each linear operator $T\colon X \to Y$ is bounded.

2.17. Let X, Y be normed spaces. A linear operator $T\colon X \to Y$ is a *coisometry* if T maps the open unit ball of X onto the open unit ball of Y.

(a) Prove that if T maps the closed unit ball of X onto the closed unit ball of Y, then T is a coisometry.
(b) Is the converse true?
(c) Show that an injective coisometry is the same thing as an isometric isomorphism.