Problems marked with "**-B**" are optional. If you solve such problems, you will earn bonus points.

3.1. Recall from the lectures that if $1 < p, q < +\infty$ and $1/p + 1/q = 1$, then there exists an isometric isomorphism $\ell^q \xrightarrow{\sim} (\ell^p)^*$. By using a similar argument, construct isometric isomorphisms **(a)** $\ell^\infty \xrightarrow{\sim} (\ell^1)^*$; **(b)** $\ell^1 \xrightarrow{\sim} (c_0)^*$. Does this approach give an isometric isomorphism $\ell^1 \cong (\ell^\infty)^*$?

3.2. Describe explicitly the duals of the following operators:
(a) the diagonal operator on ℓ^p (where $1 \leqslant p < \infty$) or on c_0;
(b) the right shift operator on ℓ^p (where $1 \leqslant p < \infty$) or on c_0;
(c) the bilateral shift operator on $\ell^p(\mathbb{Z})$ (where $1 \leqslant p < \infty$) or on $c_0(\mathbb{Z})$;
(d) the "antiderivative" operator on $L^p[0,1]$, $1 \leqslant p < \infty$ (see Problem 2.12);
(e) the Hilbert-Schmidt integral operator on $L^2(X, \mu)$ (see Problem 2.14).

3.3. (a) Prove that a linear functional on a normed space is bounded if and only if its kernel is closed. **(b)** Does (a) hold for linear operators?

3.4. Prove that for every infinite-dimensional normed space X there exists a discontinuous linear functional on X.

Hint: use the fact that each vector space has an algebraic basis (i.e., a maximal linearly independent set).

3.5-B. Prove that c_0 is not isomorphic to the dual of a normed space.

3.6. Let X be a normed space.
(a) Prove that if X^* is separable, then so is X.
(b) Is the converse true?
(c) Prove that there is no topological isomorphism between $(\ell^\infty)^*$ and ℓ^1.

3.7. Let $X = \mathbb{R}^2_p$ be the real plane endowed with the norm $\|\cdot\|_p$, and let $X_0 = \{(x, 0) : x \in \mathbb{R}\} \subset X$. Define a linear functional $f_0 \colon X_0 \to \mathbb{R}$ by $f_0(x, 0) = x$. We clearly have $\|f_0\| = 1$. Describe all linear functionals $f \colon X \to \mathbb{R}$ such that $f|_{X_0} = f_0$ and $\|f\| = 1$. (Consider all possible $p \in [1, +\infty]$.)

3.8. Let X be a normed space, and let $x_1, \ldots, x_n \in X$ be linearly independent vectors. Prove that for each $c_1, \ldots, c_n \in \mathbb{K}$ there exists $f \in X^*$ such that $f(x_i) = c_i$ for all $i = 1, \ldots, n$.

3.9. Let X and Y be normed spaces, $X_0 \subset X$ be a vector subspace, and $T_0 \colon X_0 \to Y$ be a bounded linear operator.
(a) Can T_0 be always extended to a bounded linear operator $T \colon X \to Y$?
(b) Prove that for $Y = \ell^\infty(S)$ (where S is any set) the answer to question (a) is positive.

3.10. (a) Prove that every normed space X can be isometrically embedded into $\ell^\infty(S)$ for some S. (*Hint:* let S be the unit ball of X^*.) **(b)** Prove that every separable normed space X can be isometrically embedded into ℓ^∞.

3.11. Let X be a normed space, and let $i_X \colon X \to X^{**}$ be the canonical embedding. Prove that for each operator $T \in \mathscr{B}(X, Y)$ the following diagram commutes.

$$\begin{array}{ccc} X^{**} & \xrightarrow{T^{**}} & Y^{**} \\ {\scriptstyle i_X}\uparrow & & \uparrow{\scriptstyle i_Y} \\ X & \xrightarrow{T} & Y \end{array}$$

3.12. Prove that the composition of the canonical embedding $c_0 \to (c_0)^{**}$ and the standard isomorphism $(c_0)^{**} \cong \ell^\infty$ is the inclusion of c_0 into ℓ^∞.

3.13. Prove that
 (a) a Hilbert space is reflexive;
 (b) c_0 is not reflexive;
 (c) ℓ^1 is not reflexive;
 (d) $L^1(X, \mu)$ is not reflexive (unless it is finite-dimensional);
 (e) $C[a, b]$ is not reflexive.

3.14. Let X be a normed space, and let $i_X : X \to X^{**}$ be the canonical embedding. Find a relation between the operators $i_{X^*} : X^* \to X^{***}$ and $i_X^* : X^{***} \to X^*$.

3.15. (a) Prove that a Banach space X is reflexive $\iff$ X^* is reflexive.
(b) Deduce that ℓ^1, ℓ^∞, $L^\infty[a, b]$ are not reflexive.

Definition 3.1. Let S be a set, and let $\ell^\infty(S)$ denote the space of all bounded $\mathbb{C}$-valued functions on X. A linear functional $m : \ell^\infty(S) \to \mathbb{C}$ is *positive* if $m(f) \geqslant 0$ whenever $f \geqslant 0$.

3.16-B. (a) Prove that each positive linear functional m on $\ell^\infty(S)$ is bounded, and that $\|m\| = m(1)$.
(b) Prove that a bounded linear functional m on $\ell^\infty(S)$ satisfying $\|m\| = m(1)$ is positive.

Hint. To prove (a), observe that $\langle f, g\rangle = m(f\bar{g})$ is a nonnegative hermitian form on $\ell^\infty(S)$. To prove (b), it suffices to show that if $\|m\| = m(1) = 1$, then for each $f \geqslant 0$ the number $m(f)$ belongs to every disk containing $f(S)$.

Definition 3.2. Let G be a semigroup. For each function f on G and each $x \in G$, define a function $L_x f$ by $(L_x f)(y) = f(xy)$. The semigroup G is called *amenable* if there exists a positive linear functional m on $\ell^\infty(G)$ such that $m(1) = 1$ and $m(L_x f) = m(f)$ for all $f \in \ell^\infty(G)$ and all $x \in G$. Each such m is called an *invariant mean.*

3.17-B. Prove that a finite group is amenable.

3.18-B. Prove that $\mathbb{Z}$ is amenable.

3.19-B. Prove that $\mathbb{N}$ is amenable, and that for each invariant mean m on ℓ^∞ and each convergent sequence $x = (x_n)$, $x_n \in \mathbb{C}$, we have $m(x) = \lim_{n\to\infty} x_n$.

3.20-B. Prove that the free group on 2 generators is not amenable.