Problems marked with "**-B**" are optional. If you solve such problems, you will earn bonus points.

4.1. Consider the composition of the canonical embedding $\ell^1 \to (\ell^1)^{**}$ and the standard isomorphisms $(\ell^1)^{**} \cong (\ell^\infty)^* \cong M(2^{\mathbb{N}})$ (where $M(2^{\mathbb{N}})$ is the space of all finitely additive complex measures on $2^{\mathbb{N}}$). Describe the resulting embedding of ℓ^1 into $M(2^{\mathbb{N}})$ by an explicit formula, and show that its image consists precisely of σ-additive measures.

4.2. For each of the following functionals f on $C[-1,1]$ describe the associated measure $\mu \in M[-1,1]$ and the associated distribution function $\varphi \in BV_0[-1,1]$. Find the variation $V_{-1}^1(\varphi)$ and make sure that $V_{-1}^1(\varphi) = \|f\|$.

(a) $f(x) = x(-1)$; **(b)** $f(x) = x(-1/2) - x(1/2)$; **(c)** $f(x) = \int_{-1/2}^{1} tx(t)\,dt - 2x(1)$.

4.3-B. Let (X, μ) be a measure space. Given $f \in L^1(X, \mu)$, let ν_f denote the complex measure with density f with respect to μ. Prove that $\|\nu_f\| = \|f\|_1$.

4.4. Prove that the dimension of an infinite-dimensional Banach space is uncountable.

4.5. Give an example of a normed space X and a pointwise bounded sequence (f_n) in X^* such that (f_n) is not norm bounded.

4.6. Let X, Y, Z be normed spaces.
(a) Prove that a bilinear operator $T\colon X \times Y \to Z$ is continuous if and only if there exists $C \geqslant 0$ such that $\|T(x,y)\| \leqslant C\|x\|\|y\|$ for all $x \in X$, $y \in Y$.
(b) Assume that either X or Y is complete. Prove that each separately continuous bilinear operator $X \times Y \to Z$ is continuous. (*Hint:* use the Uniform Boundedness Principle.)
(c) Does (b) hold without the completeness assumption?

4.7-B. Let G be a compact topological group, and let π be a representation of G on a Banach space X. Suppose that π is continuous in the sense that the map $G \times X \to X$, $(g, x) \mapsto \pi(g)x$, is continuous. Prove that there exists an equivalent norm $\|\cdot\|_\pi$ on X with the property that for each $g \in G$ the operator $\pi(g)$ is isometric with respect to $\|\cdot\|_\pi$. (*Hint:* the easiest way to prove this is to apply the Uniform Boundedness Principle; however, there exists a direct proof as well (see Test 1, Module 1).

4.8. **(a)** Deduce the Open Mapping Theorem from the Inverse Mapping Theorem.
(b) Deduce the Inverse Mapping Theorem from the Closed Graph Theorem.
(c)-B Deduce the Uniform Boundedness Principle from the Closed Graph Theorem.
(d)-B Deduce the Inverse Mapping Theorem for reflexive spaces from the Uniform Boundedness Principle.

Hint to (c). Given a pointwise bounded family $\{T_i\colon X \to Y \mid i \in I\}$ of bounded linear operators, construct an operator from X to the space $\ell^\infty(I, Y)$ of bounded Y-valued functions on I.

Hint to (d). To show that a bijective operator $T \in \mathscr{B}(X, Y)$ is a topological isomorphism, it suffices the prove that the preimage of the unit ball under T^* is bounded.

4.9. Give an example of a Banach space X, a normed space Y, and a bijective operator $T \in \mathscr{B}(X, Y)$ such that T^{-1} is unbounded.

4.10-B. **(a)** Give an example of a normed space X, a Banach space Y, and a bijective operator $T \in \mathscr{B}(X, Y)$ such that T^{-1} is unbounded.
(b) Give an example of an absolutely convex, absorbing set S in a Banach space such that S does not contain a 0-neighborhood.

4.11. Let X be a normed space. Prove that **(a)** every finite-dimensional subspace and **(b)** every closed subspace of finite codimension are complemented in X.

4.12-B. Prove that c_0 is not complemented in ℓ^∞.
Hint. Try the following plan:
(1) Prove that $\mathbb{N}$ can be represented as an uncountable union $\mathbb{N} = \bigcup_{i \in I} A_i$ of countable sets A_i such that $A_i \cap A_j$ is finite for $i \neq j$. (Hint: it is convenient to replace $\mathbb{N}$ by $\mathbb{Q}$).
(2) Prove that for each $f \in (\ell^\infty)^*$ that vanishes on c_0 we have $f(\chi_{A_i}) = 0$ for all but countably many $i \in I$.
(3) Prove that there is no countable set of continuous linear functionals on ℓ^∞/c_0 that separates the points of ℓ^∞/c_0.
(4) Prove that c_0 is not complemented in ℓ^∞.

4.13. (a) Suppose that a Banach space X is topologically isomorphic to Y^* for some Banach space Y. Prove that X is complemented in X^{**}.
(b)-B Do Exercise 3.5-B by using (a) and Exercise 4.12-B.

4.14. Let X and Y be Banach spaces, and let $S \in \mathscr{B}(Y^*, X^*)$. Do we always have $S = T^*$ for some $T \in \mathscr{B}(X, Y)$?

4.15. Identify $(\ell^1)^*$ with ℓ^∞ (see Exercise 3.1), and consider c_0 as a subspace of $(\ell^1)^*$. Find ${}^\perp c_0$ and $({}^\perp c_0)^\perp$.

4.16. Let X be a nonreflexive Banach space. Prove that there exists a closed vector subspace $N \subseteq X^*$ such that $N \neq ({}^\perp N)^\perp$.

4.17. Give an example of an injective operator $T \in \mathscr{B}(X, Y)$ between Banach spaces X and Y such that $\operatorname{Im} T^*$ is not dense in X^*. (*Hint:* X must be nonreflexive, see the lectures.) As a corollary, the equality $\overline{\operatorname{Im}(T^*)} = (\operatorname{Ker} T)^\perp$ can fail in the nonreflexive case.

4.18-B (*Johnson's Lemma*)**.** Let X, Y, Z be Banach spaces, $S \in \mathscr{B}(X, Y)$, $T \in \mathscr{B}(Y, Z)$, and $TS = 0$. Prove that the following conditions are equivalent:

(1) the sequence $X \xrightarrow{S} Y \xrightarrow{T} Z$ is exact, and $\operatorname{Im} T$ is closed;

(2) the sequence $Z^* \xrightarrow{T^*} Y^* \xrightarrow{S^*} X^*$ is exact, and $\operatorname{Im} S^*$ is closed.

As a corollary, a chain complex of Banach spaces is exact if and only if the dual complex is exact.

4.19-B (*Serre's Lemma*)**.** Let X, Y, Z be Banach spaces, $S \in \mathscr{B}(X, Y)$, $T \in \mathscr{B}(Y, Z)$, and $TS = 0$. Assume that S and T have closed images. Construct an isometric isomorphism $(\operatorname{Ker} T / \operatorname{Im} S)^* \cong \operatorname{Ker} S^* / \operatorname{Im} T^*$.

As a corollary, if C is a chain complex of Banach spaces such that all the maps $C_{n+1} \to C_n$ have closed images, then $H^n(C^*) \cong H_n(C)^*$.

4.20-B. Let X be a Banach space, and let $X_0 \subseteq X$ be a closed vector subspace. Prove that X is reflexive if and only if X_0 and X/X_0 are reflexive.