Functional Analysis I

Colloquium (oral midterm exam) questions

December 13, 2013

1. Bounded linear operators. Equivalent definitions. Boundedness and continuity. The norm of a bounded operator. Calculations of the norms for concrete operators.
2. Bounded linear operators. Equivalent definitions. Boundedness and continuity. Domination and equivalence of norms. The equivalence of norms on a finite-dimensional vector space.
3. Topologically injective and open operators. Isometries and coisomatries.
4. Quotients of normed spaces. The universal property of quotients. Corollaries of the universal property.
5. Banach spaces. Examples. The completeness of ℓ^p.
6. The completeness of quotients and of spaces of bounded linear operators. The extension-by-contnuity theorem.
7. The completion of a normed space. The existence and the uniqueness of the completion. The universal property and the functoriality of the completion.
8. Sesquilinear forms. Inner product spaces (definition and examples). The Cauchy-Bunyakowsky-Schwarz inequality. The norm generated by an inner product. Hilbert spaces (definition and examples).
9. The orthogonal complement of a subset of an inner product space. Basic properties of orthogonal complements. The orthogonal projection of an element onto a subspace. The uniqueness of the projection. The interpretation of the projection as the nearest point. The existence theorem for projections in a Hilbert space. The Orthogonal Complement Theorem.
10. Orthogonal and orthonormal families. Examples. Fourier coefficients and their properties. Bessel's inequality. The uniqueness of the Fourier series. Parseval's identity.
11. Orthonormal bases, total and maximal orthonormal families, relations between these notions. The existence of an orthonormal basis in a Hilbert space. The construction of an orthonormal basis in a separable inner product space via orthogonalization.
12. The characterization of Hilbert spaces as $\ell^2(I)$-spaces. The Riesz-Fischer Theorem. Hilbert dimension. Classification of Hilbert spaces.
13. The dual space of a normed space. Dual operators. Properties of taking the dual. The Riesz Representation Theorem for Hilbert spaces.
14. The dual space of a normed space. Dual operators. Properties of taking the dual. The duals of ℓ^p and $L^p(X, \mu)$.

15. The variation of a complex measure. The space of complex measures of bounded variation. The integral of a bounded measurable function w.r.t. a complex measure. The dual of the space of bounded measurable functions. Regular Borel measures. The Riesz-Markov-Kakutani Theorem (without proof).

16. Functions of bounded variation on a closed interval. Basic properties of functions of bounded variation. The distribution function of a Borel measure on the interval. The characterization of Borel measures on the interval as Lebesgue-Stieltjes measures.

17. The Riesz Representation Theorem for $C[a, b]$.

18. The Hahn-Banach Theorem. Corollaries.

19. The canonical embedding into the bidual. The functoriality of the canonical embedding. Reflexive Banach spaces. Examples.

20. Baire's Theorem. Barrels in normed spaces. The Barrel Lemma (barrels and 0-neighborhoods in Banach spaces).

21. The Uniform Boundedness Principle (the Banach-Steinhaus Theorem). Corollaries.

22. The Open Mapping Theorem, the Inverse Mapping Theorem, the Closed Graph Theorem.

23. Topological direct sums and complemented subspaces. Finite-dimensional and finite-codimensional subspaces are complemented.

24. Annihilators, preannihilators, their properties. The coincidence of the preannihilator of the annihilator with the closed linear span. Duals of subspaces and of quotients.

25. The duality between injective operators and operators with dense image. The duality between toplogically injective and surjective operators. The Closed Range Theorem (the image of an operator is closed iff the image of the dual operator is closed). The duality between kernels and cokernels.

26. The spectrum of an algebra element. Spectra of elements of $\mathbb{C}^S$, $\ell^\infty(S)$, $L^\infty(X, \mu)$. The behavior of spectra under homomorphisms. Spectrally invariant subalgebras. The Spectral Mapping Theorem for the polynomial calculus. The spectrum of the inverse element.

27. Banach algebras. Examples. Properties of the multiplicative group of a Banach algebra. The compactness of the spectrum of a Banach algebra element.

28. The resolvent function. Properties of the resolvent function. The nonemptiness of the spectrum of a Banach algebra element. The Gelfand-Mazur Theorem.

29. Spectral radius. The Beurling-Gelfand formula.

30. The spectrum of a multiplication operator. The spectrum of the shift on $\ell^2(\mathbb{Z})$. Parts of the spectrum of an operator (point spectrum, continuous spectrum, residual spectrum). Parts of the spectrum of a diagonal operator.

31. The spectrum of the dual operator. Parts of the spectrum of the dual operator. Parts of the left and right shift operators on ℓ^p (the reflexive case).