Государственное образовательное учреждение
высшего профессионального образования
Научно-Исследовательский Университет – Высшая Школа Экономики

ФАКУЛЬТЕТ МАТЕМАТИКИ

РАБОЧАЯ ПРОГРАММА ДИСЦИПЛИНЫ

Basic Representation Theory

Москва
2013

I. Syllabus of the course.

1. Mission of the course

Our mission is to teach the representation theory of a wide class of objects (groups, rings, Lie groups, Lie algebras) during one semester at a level suitable for solving problems and understanding relations with other areas in mathematics and mathematical physics.

2. Aims of the course

This is an introduction to representation theory. Upon completion of this course student will know enough to understand classical applications (in Galois Theory and Quantum Mechanics) and will be ready to take more advanced courses leading to modern problems and applications of representation theory.

In more detail, the aims are to explain

- common notions and problems of representation theory;
- various instruments for dealing with finite-dimensional representations of finite groups: intertwining operators, characters, Maschke's and Burnside's theorems;
- representations of ring and algebra in a particular case of a group ring and a group algebra.
- representations of symmetric groups and related algebraic and combinatorial constructions: Young diagrams and tableau, Young symmetrizers;
- basics of Lie groups and Lie algebras;
- representations of simplest reductive Lie algebras: SL_2, SU_2, SO_3.
- applications to Quantum Mechanics in a particular case of periodic table.

The main goal of this course is to communicate the foundations of representation theory in order to make possible further education in it. In particular, it allows the students to take more advanced courses on representation theory and related subjects as well as be able to read modern books and research papers about representation theory.

3. Novelty of methods

The main peculiarity of this course is a balance between statements in a full generality and explicit examples. Along the way, we demonstrate not only how to deduce special cases from a general statement, but also how a general theory appears as a uniform way to describe observations from solutions of concrete problems. Despite this approach is slower, it provides a deeper understanding of the subject and make the course more useful to students of various level of initial knowledge and specialization. Moreover, it makes possible to consider (at least briefly) a vast number of objects and results in a limited time, that seems to fulfill aims of an introductory course.

4. Role of the course

Representation theory describes and classifies how an abstract algebraic object (such as group, ring, associative algebra, Lie algebra) can act on vector spaces. The questions about this was arisen in the beginning of XX century in Number Theory, namely, in problems related to Galois Theory. In 30s it appeared that Quantum Mechanics is based on Representation Theory of simplest Lie groups and algebras, in particular, representations of Lie groups SO_3 and SU_2 describes the structure of atoms and predicts the electron's orbitals. Later Representation theory was applied to various branches of mathematics, first of all, to Algebraic and Differential Geometry as well as to Topology. The most important modern questions of Representation Theory are related to Quantum Field Theory and Non-Commutative Geometry, that is a generalization of Algebraic Geometry to the case of non-commutative rings.

The proposed course requires only minimal amount of prerequisites, all of them are covered by standard courses of algebra and calculus. It grants the students basic knowledge and skills in Representation Theory, that can be useful for researchers and professors in mathematics and mathematical physics, as well as just expand a scope for a specialist in a different area.

II. Essence of the course

1. History of the course

This course is based on earlier courses given by O.Sheinman, E.Smirnov, V.Ivanov and myself in the Independent University of Moscow as well as on literature listed below. This program is written to make the course suitable both for "Math in Moscow" program in the Independent University of Moscow and for the magister program "Mathematics" (in English) in NRU HSE.

The course consists of three topics that are usually discussed in different courses: representations of finite group, representations of Lie groups and algebras, quantum mechanics with Bohr's model of an atom. The first topic is relatively easy and traditional, it is partially contained in a general course of algebra for mathematicians in Moscow State University, Independent University of Moscow and Mathematical Department of NRU HSE. The second topic is usually given as a separate advanced course that lasts at least one semester. Despite the connections of the third topic to the first two, it is usually given separately and mostly for prospective specialists in physics.

There are two reasons for the intensity of this course. Students of the magister program need a concise introduction to their specialty in order to obtain a result and write a qualifying paper at the end of the second year. Students of the "Math in Moscow" program usually can not stay in Moscow for more than a semester. Nevertheless, all the students obtain enough knowledge to continue with more advanced course or other ways to study areas of mathematics and mathematical physics that requires representation theory.

2. Thematic plan

название темы	количество часов		
	лек	упр	сам
Subject of representation theory.	2	2	2
Complete reducibility of finite group representations.	2	2	2
Characters of finite group representations.	4	4	4
Relation to representations of rings and algebras.	2	2	2
Representations of the symmetric group.	4	4	4
Compact groups, Lie groups.	4	4	4
Lie algebras, relation to Lie groups.	2	2	2
Representations of Lie groups and algebras.	2	2	2
Reductive Lie groups.	2	2	2
Representation of groups SL_2, SO_3, SU_2.	2	2	2
Applications to quantum mechanics.	2	2	2
итого	28	28	28

III. Details of the program

Subject of representation theory. Definition of a group representation, examples. Subrepresentation and quotient representation. Direct sum of representations, irreducible and indecomposable representation. Homomorphism of representations (intertwining operator), isomorphism of representations. Schur's lemma.

Complete reducibility of finite group representations. Maschke's theorem on complete reducibility of finite group representations. Different proofs using invariant projectors and invariant positively defined forms. Examples of reducible but indecomposable representations for infinite groups as well as over fields with positive characteristic. Uniqueness of a decomposition into irreducible representations, multiplicities of irreducible subrepresentations. Intertwining operators from regular representation, a decomposition of a regular representation. The Burnside's formula and its analog for an arbitrary field. Examples: representations of an abelian group and the dihedron group.

Characters of finite group representations. Definition of a character of a complex representation as a function on a group. Character of the dual representation, the direct sum of representations and the tensor product of representations. Scalar product of functions on a group, orthogonality of matrix elements of irreducible representations, orthonormality of characters of irreducible representations. A character based criteria of irreducibility of a representation and when two representations are isomorphic. A formula for multiplicity of an irreducible subrepresentation. Completeness of characters in the space of central functions on a group, equality of number of complex irreducible representations up to an isomorphism and number of conjugation classes.

Relation to representations of rings and algebras. Definitions of a representation of a ring and a representation of an algebra. A group ring and g group algebra, a relation of their representation to representation of the group. Structure of group algebra over algebraically closed fields. Relation to the regular representation.

Representations of the symmetric group. Young diagram, relation to conjugation classes of symmetric group. A construction of representations based on Young symmetrizators in the group algebra. A decomposition of a representation, restricted from S_n onto S_{n-1} (without proof). Examples.

Compact groups, Lie groups. Topological groups. Proof of Maschke's theorem and character theory for compact groups. Structure of smooth manifold. A definition and examples of Lie groups, matrix Lie groups. Homomorphisms of Lie groups, Lie subgroups. A dense cable of torus as a subgroup which is not a Lie subgroup. The connected component of the unit as the maximal connected normal Lie subgroup. Any connected Lie group is generated by any open vicinity of unit.

Lie algebras, relation to Lie groups. An abstract definition of Lie algebra, examples. The tangent space of a Lie group in the unit as a Lie algebra. An exponential map. Homomorphisms of Lie algebras. Differential of a Lie group homomorphism as a tangent Lie algebra homomorphism.

Representations of Lie groups and algebras. A definition. A construction of a tangent Lie algebra representation from a Lie group representation. A relation of subrepresentations of a Lie group with subrepresentations of a tangent Lie algebra. Examples: adjoint, coadjoint and tautological representations. Infinite-dimensional examples: representations in functions on a homogeneous space.

Reductive Lie groups. A definition of a reductive Lie group. Real forms of a complex Lie groups and algebras. Reductive groups as Lie groups with compact real form. Complete reducibility of finite-dimensional complex representations of reductive Lie group (Weyl's unitary trick). Examples: classical groups SL_n, SO_n, SP_n and their compact analogs.

Representations of groups SL_2, SO_3, SU_2 A homomorphism $SU_2 \to SO_3$. Representations of a Lie algebra $\mathfrak{sl}_2$. Highest weight representations of $\mathfrak{sl}_2$. Uniqueness of an irreducible representation with a given highest weight, its dimension. A decomposition of the tensor product of irreducible representations. Infinite-dimensional representations, Verma modules. Examples of reducible indecomposable infinite-dimensional representations.

Applications to quantum mechanics. The Quantum Mechanics settings. Bohr's model of an atom. The periodic table form the point of view of representation theory of the group SO_3.

IV. Textbooks and other references.

In English

1. Fulton, William; Harris, Joe *Representation theory. A first course*, Graduate Texts in Mathematics, Readings in Mathematics, 129, New York: Springer-Verlag, 1991

2. J.E. Humphreys *Introduction to Lie Algebras and Representation Theory*, Springer 1973

3. A.A.Kirillov, J.Bernstein, V.Arnold, *Representation Theory*, Abdus Salam School of Mathematical Sciences, 2009

4. Sheinman, Oleg K. *Basic Representation Theory*, Moscow: MCCME, 2007.

5. Serre, Jean-Pierre, *Linear Representations of Finite Groups*, Springer-Verlag, 1977.

6. Vinberg, Èrnest B. *A Course in Algebra*, Graduate Studies in Mathematics, 56, AMS, 2003.

In Russian

1. Э. Б. Винберг, *Курс алгебры*, М.: Факториал, 1999.

2. Э. Б. Винберг, *Линейные представления групп*, М.: Наука, 1985.

3. И. М. Парамонова, О. К. Шейнман, *Задачи семинара "Алгебры Ли и их приложения"*, М.: МЦНМО, 2004.

4. О. К. Шейнман, *Основы теории представлений*, М.: МЦНМО, 2004.

V. Grading procedure

The procedure described here works both for "Math in Moscow" program and for magister program at the department of mathematics.

A homework is formed each week from several problems not discussed during the seminar. The student are to submit next week a written solutions and we grade it to obtain a mark. A total of these marks forms 40 percents of the resulting mark. A sample of homeworks is attached to this program.

According to the regulations of NRU HSE, in the middle of the course there is a written classwork which is called *midterm exam* in "Math in Moscow". A mark for this work forms 20 percents of the resulting mark.

Just after the course there is a written exam, which is called *final exam* in "Math in Moscow". A mark for this work forms the remaining 40 percents of the resulting mark.

The midterm exam and the final exam lasts 1.5 and 3 hours respectively. Students are allowed to use their class notes and common materials during these exams.

VI. Printed materials.

A sample of homeworks and exams, prepared by me for a similar course in 2002-2003 year, as well as several materials of this year are attached to this program.