EXPERIMENTAL MATHEMATICS
Topics for projects
Deadline: 24:00 June 10, 2014

Choose a topic and complete your project in *Mathematica*. For more topics to choose from please contact Professor Maxim Kazarian kazarian@mccme.ru. For answers to your questions and getting comments, please address Maxim Kazarian kazarian@mccme.ru; Leonid Rybnikov leo.rybnikov@gmail.com; Nikita Markarian nikita.markarian@gmail.com; Sergei Loktev S.Loktev@gmail.com; Toshiro Kuwabara toshiro.kuwa@gmail.com; Sergei Lando lando@hse.ru, depending on the group the topic belongs to.

(1) **COMBINATORIAL GEOMETRY** N. Markaryan, L. Rybnikov

(a) **Permutohedron.** A 3-dimensional permutohedron is the convex hull of points

$$(a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)}, a_{\sigma(4)}),$$

where (a_1, a_2, a_3, a_4) is a point in $\mathbb{R}^4$, and σ runs through the group S_4 of all permutations on 4 elements. Permutohedron lies in the hyperplane $x_1+x_2+x_3+x_4 = \text{const}$ (hence, it is indeed a 3-dimensional polytope). Program a manipulator that draws the permutohedron for different choices of the point (a_1, a_2, a_3, a_4).

(b) **Newton polygons.** Program a manipulator that draws the Newton polygon for each Laurent polynomial in two variables from a large collection of polynomials (e.g. for all polynomials whose Newton polygons lie inside the square $[-100, 100] \times [-100, 100]$).

(c) **Roots.** Program a manipulator that draws Dynkin diagrams and all simple roots for each of the rank 2 and rank 3 root systems (that is, for A_2, B_2, F_2, G_2, A_3, ...).

(d) **Convex polytopes.** A convex polytope in $\mathbb{R}^n$ can be defined either as the convex hull of points or as the intersection of halfspaces (a *halfspace* is the region defined by a linear inequality). Write linear inequalities defining a polytope if its vertices are given. In the other direction, find the vertices of the polytope defined by given inequalities.

(e) **Mitosis.** Mitosis is an operation on $(n \times n)$–tables whose cells can be filled either with sign "+" or left empty. For every $i \in \{1, \ldots, n-1\}$, the mitosis operation M_i assigns to each table T several new tables as follows. If cell $(i, 1)$ is empty, then $M_i(T) = T$. If cell $(i, 1)$ contains $+$, then find the minimal j such that cell (i, j) is empty. Let $K \subset \{1, \ldots, j-1\}$ be the set of all $k < j$ such that cell $(i+1, k)$ is empty. For each $k \in K$, construct the table T_k by erasing $+$ in cell (i, k) and shifting all $+$ in cells (i, l) (if any) by one cell down for all $l \in K \cap \{1, \ldots, k\}$ (in particular, T_k coincides with T in all rows except for the ith and $(i+1)$st one). Define $M_i(T)$ as the set of all tables T_k, where $k \in K$.
Write a program that finds the set of tables $M_i(T)$ for every table T.

(f) **Voronoi diagrams.** Consider a finite set X of points in the plane. The *Voronoi cell* of an element $x \in X$ is defined as the set of points in the plane whose distance to x is smaller than the distance to any other element of X. *Voronoi diagram* is a decomposition of the plane into Voronoi cells of the points of X. Program a manipulator that allows the user to choose his own points or choose random sets of points and produce their Voronoi diagram.

(g) **Wallpaper groups.** Draw a fundamental domain of a wallpaper group. The user should be able to change the shape of a fundamental domain and choose any of the 17 wallpaper groups.

(2) **DYNAMICAL SYSTEMS** N. Markaryan, L. Rybnikov

(a) **Billiards.** Program a manipulator that draws billiard trajectories for different polygons and ellipses. The user should be able to choose a polygon or ellipse, the initial position and the number of iterations.

(b) **Vector fields and limit cycles.** For a vector field depending on a parameter, draw the limit cycles, neighboring trajectories and the vector field itself. For instance, take the family of van der Pol equations

$$\begin{array}{l} \frac{dx}{dt} = \varepsilon(-(x^3/3) + x - y) \\ \frac{dy}{dt} = \frac{x}{\epsilon} \end{array}.$$

The user should be able to change the parameter ε.

(c) **Geometric dynamics.** Consider the following dynamical system. To a triangle assign its orthotriangle. Realize this dynamical system by a manipulator. Triangles should be dilated to keep their size roughly the same. What does dynamics look like?

(d) **String.** Draw an animation showing motion of a bounded string whose right endpoint is fixed and the left endpoint is perturbed (i.e. someone pulls it according to a certain law). Mathematically, this means solving the wave equation $u_{tt} = a^2 u_{xx}$ with boundary conditions $u_t(x, 0) = u(x, 0) = 0$ if $x \in [0, 1]$ (string is not perturbed at the zero time), $u(0, t) = \mu(t)$ if $t > 0$, where $\mu(t)$ is a given function (the law of perturbation), $u(1, t) = 0$ (the right endpoint is fixed).

(e) **Attractors for rational functions.** Consider the family of quadratic rational functions depending on a complex parameter a

$$f_a(x) = \frac{a}{x^2 + 2x}.$$

Points 0 and ∞ form a superattracting cycle for the iterations of these functions. Color by yellow and blue the points that tend to this cycle under iterations (the color of a point depends on the parity of iterations that bring the point close to 0).

(3) **DIFFERENTIAL GEOMETRY** M. Kazarian, S. Lando

(a) **Geodesics on ellipsoid.** Draw geodesics on the 3-dimensional ellipsoid

$$\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 1.$$

There should be an opportunity to choose values of the parameters a, b, c.

(b) **Doughnut and pretzel.** Program a manipulator that draws the surfaces of a torus and a sphere with 2 handles. The user should be able to change the thickness of handles and rotation angle.

(c) **Envelope of lines.** Write a program that draws a family of lines and its envelope. The user should be able to change the parameters of a family.

(d) **The caustic of ellipsoid.** Write a program that draws the caustic of the 3-dimensional ellipsoid

$$\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 1.$$

There should be an opportunity to choose values of the parameters a, b, c.

(4) **GROUPS** T. Kuwabara, S. Loktev

(a) **Abelian group.** Recall that every finitely generated Abelian group can be uniquely represented as $\mathbb{Z}^r \oplus \mathbb{Z}/n_1\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/n_l\mathbb{Z}$ where $n_1|\ldots|n_l$. The numbers $n_1,\ldots,n_l$ are called *invariant factors.* Write a program computing invariant factors for a finitely generated Abelian group with 10 generators and 5 relations (the group may be encoded by a 10×5 matrix with integer entries).

(b) **Irreducible polynomials.** Output the list of all irreducible polynomials of degree at most n over the field Z/pZ.

(c) **Symmetric functions.** Express a basis in symmetric functions in terms of another basis (standard ones being m_λ, e_λ, Newton polynomials p_λ, Schur polynomials s_λ, Hall–Littlewood polynomials H_λ).

(d) **Characters.** Computation of the characters of representations of symmetric groups.

(5) **ALGEBRAIC GEOMETRY** S. Loktev, L. Rybnikov

(a) **Conics.** Program a manipulator that draws the family of conics passing through four given points on the plane and all conics tangent to four given lines. The user should be able to choose the positions of points and lines. Ideally, for any choice of $k = 0,\ldots,4$ points and $4 - k$ lines, the manipulator should draw all conics passing through chosen points and tangent to chosen lines.

(b) **Poncelet problem.** If there exists a triangle inscribed in a given conic and superscribed around another given conic then there are infinitely many such triangles (any point on the first conic can be a vertex of such a triangle). The same holds for a polygon with any number of vertices. Draw the corresponding pictures so that the user will be able to change conics and polygons.

(c) **Cubic curves.** Program addition of points on cubic curves. The user should be able to choose a cubic and two points on the cubic from a sufficiently large collections of cubics and of rational points on them.

(d) **Amoebas.** Let C_t be the complex plane curve given by the polynomial equation $\sum_{(i,j)\in\Delta} t^{\alpha_{i,j}} c_{i,j} x^i y^j = 0$ with $c_{i,j} \in \mathbb{C}$, $\alpha_{i,j} \in \mathbb{R}$, $t \in \mathbb{R}$, and Δ a finite set of indices. The amoeba of C_t is its image under the map $\mathbb{C}^2 \to \mathbb{R}^2$, $(x, y) \mapsto (\log_t |x|, \log_t |y|)$. Program a manipulator that draws the amoebas for given curves of small degrees depending on the parameter t, and the limiting image as $t \to \infty$.

(6) **FUNCTIONS** M. Kazarian, S. Lando

(a) **Pade and Lagrange Approximations.** For a given function (e.g. sine), draw the plots of rational and polynomial approximations of this function depending on the interpolation nodes. Program a manipulator that allows the user to move interpolation nodes.

(b) **Degenerations of critical points.** A critical point of a function f is called degenerate if the second differential at this point is a degenerate quadratic point. For two degree 4 polynomials P and Q in two variables, find the values of the parameter t such that the function $P+tQ$ has a degenerate critical point. Program a manipulator that draws the critical points of the function $P+tQ$ for each values of t and writes the indices of these critical points (index = number of minuses in the diagonalization of a quadratic form).

(c) **Discriminant.** Draw the caustic and the Maxwell stratum in the space of polynomials $x^5+a_2x^3+a_3x^2+a_4x$ so that the geometry of the two around the origin could be understood in detail.

(d) **Roots of polynomials.** Using the Sturm system, compute the number of roots of a given polynomial on a given interval.

(7) **GRAPHS** M. Kazarian, S. Lando

(a) **Graph invariants.** Write a program allowing a user to draw a graph, whose output are various graph characteristics (number of vertices, number of edges, chromatic polynomial, and so on).

(b) **Intersection graphs.** Write a program allowing a user to draw a connected graph, checking whether the graph is the intersection graph of a chord diagram, and drawing all such diagrams.

(c) **Graphs on surfaces.** Write a program allowing a user to draw a connected graph, computing the minimal genus of a surface into which this graph is embeddable, and drawing such an embedding.

(d) **Automorphism group.** Find all pairs of graphs with 7 vertices having isomorphic automorphism groups.

(e) **Cartographic group.** Find all pairs of embedded graphs with 7 vertices having isomorphic cartographic groups.

(8) **POWER SERIES** M. Kazarian, S. Lando

(a) **KdV equation** The Korteweg–de Vries hierarchy is an infinite system of PDE's of the form $\partial_k u = P_k(u,u',u'',\dots)$ for the unknown function $u(t_0,t_1,\dots)$,where $\partial_k = \frac{\partial}{\partial t_k}$, $' = \frac{\partial}{\partial t_0}$ and where the sequence of polynomials P_k is determined by the equality $(2k+3)P'_{k+1} = P'_0P_k + 2P_0P'_k + \frac{1}{4}P'''_k$. Write a program allowing one to compute various mixed partial derivatives $\partial_{k_1}\partial_{k_2}\dots u$ as polynomials in $u,u',u'',\dots$ and to check the integrability of the hierarchy showing that the mixed partial derivatives taken in different order are equal.

(b) **Power series inversion.** Check experimentally the summation over graphs formula for the coefficients of the inverse function. If $y = x + a_2\frac{x^2}{2!} + a_3\frac{x^3}{3!} + \dots$, then $x = y + b_2y^2 + b_3y^3 + \dots$ where b_n is the sum of contributions of all possible rooted trees with n leaves and no vertices of valency 2. The contribution of a tree with internal vertices of valencies $k_1,\dots,k_\ell$ is equal to the monomial $(-1)^\ell\prod_{i=1}^{\ell} a_{k_i+1}$ divided by the order of the automorphism group of the tree.

(c) **Ribbon graphs enumeration.** The coefficient $\mathcal{N}_{g,n}(\mu) = \mathcal{N}_{g,n}(\mu_1, \ldots, \mu_n)$ of an infinite formal power series $F(p_1, p_2, \ldots; \hbar) = \sum_{g,n} \frac{\hbar^g}{n!} \sum_{\mu=(\mu_1,\ldots,\mu_n)} \mathcal{N}_{g,n}(\mu)\, p_{\mu_1} \ldots p_{\mu_n}$ is defined as the number of ribbon graphs of genus g with n marked vertices of prescribed valencies $\mu_1, \ldots, \mu_n$ and counted with the weights inverse to the orders of the automorphism group. Compute several initial terms of the series and check that the Kadomtsev—Petviashvili equation holds for this function, that is, $F_{2,2} = F_{1,3} - \frac{1}{2}F_{1,1}^2 - \frac{\hbar}{12}F_{1,1,1,1}$.

(9) **MISCELLANEOUS** T. Kuwabara, N. Markaryan

(a) **Sequence 1 11 21 1211.** Write a function generating the following sequence

$$1 \to 11 \to 21 \to 1211 \to 111221 \to 312211 \to \ldots$$

Is this sequence periodic? If yes, what is its period? If not, estimate the average growth of its element lengths.

(b) **Arbelos.** Program a manipulator that draws Archimedes arbelos (i.e. the plane figure bounded by 3 semicircles with diameters $[(0,0),(a,0)]$, $[(a,0),(1,0)]$, $[(0,0),(1,0)]$). Inscribe the circle into the arbelos, draw a radius of the circle and the perpendicular from the center of the circle to the diameter $[(0,0),(1,0)]$. The user should be able to change the parameter a (and notice that the length of the perpendicular is always equal to twice the length of the radius).

(c) **Quadratic irrationalities.** Write a program that finds the period of the continued fraction of a given quadratic irrational number.

(d) **Schubert varieties.** Which Schubert varieties are smooth? With every permutation one can associate an algebraic variety called *Schubert variety*. Not every Schubert variety is smooth. There is a simple smoothness criterion. Consider a permutation p. If the sequence $p(a)p(b)p(c)p(d)$ coincides with the sequence $d'b'c'a'$ or with the sequence $c'd'a'b'$ for some $a < b < c < d$ and $a' < b' < c' < d'$, then the corresponding Schubert variety is singular (otherwise, it is smooth).
Find all permutations on 4 and 5 elements such that the corresponding Schubert varieties are smooth. Compute the number of singular Schubert varieties for permutations on 6 elements.

(e) **Sudoku.** Write a program for solving Sudoku puzzle.

(f) **Polygon section.** Program a manipulator drawing all possible straight lines that cut off a domain of prescribed area from a given convex polygon on the plane.