STUDENT PROJECTS 2014-2015

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1. First and second year students

Background The group $\mathrm{SL}(n, \mathbb{R})$ acts transitively on the space P of $n \times n$ positive definite matrices of determinant 1 via $X \mapsto gXg^T$ for $g \in \mathrm{GL}(n, \mathbb{R}), X \in \mathrm{P}$. The stabiliser of the identity matrix Id is the special orthogonal group $\mathrm{SO}(n, \mathbb{R})$, and so the space P is identified with coset space $\mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n, \mathbb{R})$. The subgroup $\mathrm{SL}(n, \mathbb{Z}) \subset \mathrm{SL}(n, \mathbb{R})$ acts discretely on P and has various nice fundamental domains $D \subset P$. Siegel showed that the (suitably normalised) volume of a fundamental domain is

$$\mathrm{vol}(D) = \zeta(2)\zeta(3) \cdots \zeta(n).$$

Project 1 Survey the background for Siegel's result and write out the details of his argument or its variations.

Main reference: Siegel's Lectures on the Geometry of Numbers.

Project 2 After a suitable change of coordinates, one can replace the space P of positive definite matrices of determinant 1 with the space Q of positive definite matrices of trace 1 ($X \mapsto X/\mathrm{tr(X)}$). The space Q has the advantage that it is bounded in the space of all symmetric matrices and its closure is the space $\overline{Q}$ of positive semidefinite matrices of trace 1. There is a method of Selberg for constructing fundamental domains inside Q for discrete subgroups $\Gamma \subset \mathrm{SL}(n, \mathbb{R})$ of finite covolume. Such fundamental domains are given as the interiors of the convex hull of finitely points in $\overline{Q}$. Give a useful sufficient criterion for a convex hull in $\overline{Q}$ to have finite volume (with respect to the invariant measure on Q). The solution to this problem in the analogous but easier case of hyperbolic space is due to Igor Rivin.

2. Third and fourth year students

Background Moduli spaces of vector bundles (or complexes thereof) on a Calabi-Yau threefold are known to have a very interesting local structure. Namely, it is known that any point E in such a moduli space has a neighbourhood isomorphic to the critical locus of a regular function $W : U \to \mathbb{A}^1$ on a smooth variety U. Using W, one can associate to the point E a very interesting invariant: the so-called space of vanishing cycles for W, together with the extra structure of a mixed Hodge structure (when working over the complex numbers). Perhaps all known explicit examples of the function W come from quivers and it is very desirable to have explicit examples coming from algebraic geometry. In principle, one should be able to compute a Taylor expansion of W using deformation theory, and this would be interesting to do in the case of a point on a Hilbert scheme of zero-dimenional schemes or curves on a Calabi-Yau threefold. This question is local on the normal sheaf of the subscheme of the Calabi-Yau threefold and doesn't depend on the ambient threefold, making the computation simpler.

Project 1 Given an interesting family of zero-dimensional subschemes or curves $Z \subset X$ in a Calabi-Yau threefold, compute $\mathrm{Ext}^*_{\mathrm{X}}(\mathcal{O}_{\mathrm{Z}}, \mathcal{O}_{\mathrm{Z}})$ as an algebra, with explicit dependence on Z.

Project 2 Under suitable simplying hypotheses, compute an A_∞ minimal model structure on $\mathrm{Ext}^*_{\mathrm{X}}(\mathcal{O}_{\mathrm{Z}}, \mathcal{O}_{\mathrm{Z}})$, with explicit dependence on Z. To see what the possibilities can be, one should compute Hochschild cohomology of $\mathrm{Ext}^*_{\mathrm{X}}(\mathcal{O}_{\mathrm{Z}}, \mathcal{O}_{\mathrm{Z}})$ as differential graded algebra. Given an explicit minimal model, one can assemble the higher multiplications of the model into the Taylor expansion of W, from which one can begin to understand the associated invariants on vanishing cycles.

Main reference: Section 3.3 of Kontsevich-Soibelman's Stability structures, motivic Donaldson-Thomas invariants and cluster transformations.