Exercises 07.09.2015

1. Let $A = \mathsf{k}[G]$, and $\pi^*\colon A \to A \otimes A$, $\iota^*\colon A \to A$. Let also γ^* be the composition $A \to \mathsf{k} \to A$ where the first morphism is $f \mapsto f(e)$. Show that the group axioms for G are equivalent to the following Hopf algebra axioms for A: (a) $(\pi^* \otimes \mathrm{Id}) \circ \pi^* = (\mathrm{Id} \otimes \pi^*) \circ \pi^*$; (b) $m \circ (\iota^* \otimes \mathrm{Id}) \circ \pi^* = m \circ (\mathrm{Id} \otimes \iota^*) \circ \pi^* = \mathrm{Id}$ where $m\colon A \otimes A \to A$ is the multiplication; (c) $m \circ (\gamma^* \otimes \mathrm{Id}) \circ \pi^* = m \circ (\mathrm{Id} \otimes \gamma^*) \circ \pi^* = \mathrm{Id}$.

2. Let $\phi\colon G \to H$ be a homomorphism of linear algebraic groups, and $\phi^*\colon \mathsf{k}[H] \to \mathsf{k}[G]$ the corresponding homomorphism of coordinate rings. Let $I_e \subset \mathsf{k}[H]$ be the ideal of functions vanishing at the identity element $e \in H$. Set $I = \phi^*(I_e)\mathsf{k}[G]$. Check that $A = \mathsf{k}[G]/I$ defines an affine group scheme (but not necessarily reduced!). It will be denoted the scheme-theoretic kernel of ϕ.

3. An affine group scheme is called finite if the corresponding coordinate algebra A is finite dimensional. It is called commutative if the image of $\pi^*\colon A \to A \otimes A$ is invariant under the involution $f \otimes g \mapsto g \otimes f$. Check that if A corresponds to a commutative finite group scheme G, then the dual vector space A^* also has a structure of a commutative Hopf algebra, so that it corresponds to a finite group scheme denoted $G^\vee$.

4. Assume that k has characteristic $p > 0$. Let G be the scheme-theoretic kernel of the map $\mathbb{G}_m \to \mathbb{G}_m$, $t \mapsto t^p$. Describe $G^\vee$.

5. Compute the group of algebraic group automorphisms of $\mathbb{G}_a$.

6. Prove that the only algebraic group automorphisms of $\mathbb{G}_m$ are $x \mapsto x$ and $x \mapsto x^{-1}$.

7. Prove that a prescheme (a not necessarily separable scheme) G with an algebraic group structure is necessarily separable, so that the separability axiom in the definition of an algebraic group is redundant.