2015-11-16 Introduction to number theory Problems

References:
Kato-Kurokawa-Saito, "Number Theory 2: Introduction to Class Field Theory", Section 6.2, American Mathematical Society.
J.S.Milne's lecture note "Algebraic Number Theory" available at
http://www.jmilne.org/math/CourseNotes/ant.html

1. Let K be a field and $\nu : K \to \mathbb{Z}$ be a discrete valuation. Show that

$$\{a \in K \mid \nu(a) \geq 0\}$$

is a subring of K.

2. Let $K = \mathbb{C}(T)$. Construct a discrete valuation $\nu : K^\times \to \mathbb{Z}$ such that $\nu(T-2) = 1$.

3. Given a discrete valuation ring A, we equip A with the topology using the valuation. We saw in the lecture that $\mathbb{Z}_p$ is compact for this topology.
3.1. Show that $\mathbb{C}[[T]]$ is not compact.
3.2. Show that $\mathbb{Z}_{(p)} = \{\frac{a}{b} \mid a, b \in \mathbb{Z}, p \nmid b\}$ is not compact.

4. Let A be a discrete valuation ring. Let $\nu : \mathrm{Frac}(A) \to \mathbb{Z}$ be the discrete valuation whose valuation ring is A. Let $\alpha \in A$ be an element such that $\nu(\alpha) = 1$.
4.1. Let $\mathfrak{a} \subset A$ be an ideal. Set $n = \min\{\nu(x) \mid x \in \mathfrak{a}\}$. Show that $\mathfrak{a} = (\alpha^n)$.
4.2. Show that the set of units $A^\times$ equals $\{a \in \mathrm{Frac}(A) \mid \nu(a) = 0\}$.
4.3. Show that (α) and (0) are the only prime ideals of A.

5. Let A be a Dedekind domain. Suppose that there are exactly two prime ideeals (0) and $\mathfrak{p}$. Show that A is the discrete valuation ring for the valuation (on $\mathrm{Frac}(A)$) given by $\mathrm{ord}_{\mathfrak{p}}$.

6. Let K be a field and ν be a discrete valuation. Show that, if $\nu(x) > \nu(y)$ for $x, y \in K$, then $\nu(x+y) = \nu(y)$ holds.

7. Let A be a discrete valuation ring. Let $\wp \subset A$ be the nonzero prime ideal. Suppose $A/\wp$ is a finite field. Show that $A/\wp^n$ is a finite ring for each $n \geq 1$.

8. Let A be a discrete valuation ring. Let $\wp \subset A$ denote the nonzero prime ideal. Let $K = \mathrm{Frac}(A)$ and let $k = A/\wp$ be the residue field. Show that $\mathrm{char}K \leq \mathrm{char}k$.