2015-10-12 Problems

Introduction to Number Theory

References:
"Number Theory 1: Fermat's dream" Kato, Kurokawa, Saito, Chapter 3.

1. In this exercise, we prove the formula due to Bernoulli and Seki.
1.1. Let $D : \mathbb{C}[x] \to \mathbb{C}[x]$ denote the linear operator

$$D(f(x)) = \frac{d}{dx}(f(x)).$$

Set

$$e^D = \sum_{n=0}^{\infty} \frac{D^n}{n!}.$$

1.2. Show that $e^D(f(x)) = f(x+1)$ holds for any $f(x) \in \mathbb{C}[x]$.
1.3. Show that

$$D(f(x)) = e^D \sum_{n=0}^{\infty} \frac{B_n}{n!}(D^n(f(x))) - \sum_{n=0}^{\infty} \frac{B_n}{n!}(D^n(f(x)))$$

holds for any $f(x) \in \mathbb{C}[x]$. Here $D^n(f(x)) = D(D(\dots(D(f(x)))$ (D appearing n-times).
1.4. Show that

$$rx^{r-1} = B_r(x+1) - B_r(x).$$

(Apply 1.2 above to x^r.)
1.5. Deduce the following formula:

$$\sum_{n=0}^{x-1} n^{r-1} = \frac{1}{r}(B_r(x) - B_r).$$

2.1. Substitute $x = i(= \sqrt{-1})$ in the formula (Theorem 2.1, 12.10.2015) for $r = 1$ and show

$$\sum_{n \in \mathbb{Z}} \frac{1}{n^2+1} = \pi \cdot \frac{e^{2\pi}+1}{e^{2\pi}-1}$$

2.2. Substitute $x = i(= \sqrt{-1})$ in the formula for $r = 2$ (Theorem 2.1, 12.10.2015) and show

$$\sum_{n \in \mathbb{Z}} \frac{1}{(n^2+1)^2} = \frac{\frac{\pi}{2}e^{4\pi} + 2\pi^2 e^{2\pi} - \frac{\pi}{2}}{(e^{2\pi}-1)^2}.$$

3.1. Show

$$\zeta_{\equiv a(N)}(0) = -\frac{a}{N} + \frac{1}{2}.$$

3.2. Show

$$\zeta_{\equiv a(N)}(-1) = -\frac{a^2}{2N} + \frac{a}{2} - \frac{N}{12}.$$

3.3. Show

$$\zeta_{\equiv a(N)}(-2) = -\frac{a^3}{3N} + \frac{a^2}{2} - \frac{Na}{6}.$$

Some analytic properties

$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$.

$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$. ($\chi$ is a Dirichlet character.)

$\zeta_{\equiv a(N)}(s) = \sum_{n=1, n\equiv a(N)}^{\infty} \frac{1}{n^s}$.

$\zeta(s,x) = \sum_{n=1}^{\infty} \frac{1}{(n+x)^s}$. ($x \in \mathbb{R}, x > 0$.)

Proposition 1 (Re$(s) > 1$)**.**

1. *The series above converge absolutey for s satisfying* $\mathrm{Re}(s) > 1$,
2. *They are holomorphic in this domain.*

Proposition 2 ($s \in \mathbb{C}$)**.**

1. *They have analytic continuation to the entire complex plane $s \in \mathbb{C}$.*
2. *They are meromorphic functions there ($s \in \mathbb{C}$).*
3. *They are holomorphic in $s \neq 1$.*
4. $\lim_{s\to 1}(s-1)\zeta(s) = 1$,
5. $\lim_{s\to 1}(s-1)\zeta_{\equiv a(N)}(s) = \frac{1}{N}$,
6. $\lim_{s\to 1}(s-1)\zeta(s,x) = 1$,

Proposition 3 ($L(s,\chi)$ for nontrivial χ)**.** *Suppose the image of $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ is not $\{1\}$ (we say χ is nontrivial). Then*

1. *the defining series of $L(s,\chi)$ converge (the sum taken in the order $n = 1, 2, 3, \ldots$) for s satisfying* $\mathrm{Re}(s) > 0$.
2. *It is holomorphic in* $\mathrm{Re}(s) > 0$.
3. *The analytic continuation is holomorphic in the entire complex plane.*

The values at some integers of zeta functions

Theorem 1. *The values of the Riemann zeta:*

1. *Let r be a positive even integer. Then*

$$\zeta(r) = \frac{1}{(r-1)!} \cdot \frac{1}{2^r - 1} \cdot (2\pi i)^r \cdot \frac{1}{2} \cdot h_r(-1).$$

2. $$\zeta(0) = -\frac{1}{2}.$$

3. *Let $r \geq 2$ be an integer. Then*

$$\zeta(1-r) = -\frac{1}{r} B_r.$$

Theorem 2. *The values of the Dirichlet L:*
Let $N \geq 2$. Let χ be a Dirichlet character modulo N.

1. *Let $r \in \mathbb{N}$ and assume $\chi(-1) = (-1)^r$. Set $\zeta_N = e^{2\pi i/N}$. Then*

$$L(r, \chi) = \frac{1}{(r-1)!} \cdot \left(-\frac{2\pi i}{N}\right)^r \cdot \frac{1}{2} \cdot \sum_{a \in (\mathbb{Z}/N\mathbb{Z})^\times} \chi(a) h_r(\zeta_N^a).$$

2. *Assume that χ is nontrivial. Then*

$$L(0, \chi) = -\frac{1}{N} \sum_{a=1}^{N} a\chi(a).$$

Theorem 3. *The values of the Hurwitz zeta:*
Let $r \in \mathbb{N}$ and $x \in \mathbb{R}, x > 0$. Then

$$\zeta(1-r, x) = -\frac{1}{r} B_r(x).$$

Theorem 4. *The values of the partial zeta:*
Let $r, N, a \in \mathbb{N}$ with $1 \leq a \leq N$.

$$\zeta_{\equiv a(N)}(1-r) = -\frac{1}{r} N^{r-1} B_r(\frac{a}{N}).$$

Set

$$h_1(t) = \frac{1+t}{2(1-t)}, \ h_r(t) = \left(t\frac{d}{dt}\right)^{r-1} (h_1(t)). \ (r \geq 1)$$

Proposition 4. *Let $x \in \mathbb{C}, x \notin \mathbb{Z}, t = e^{2\pi i x}$. Then*

$$h_1(t) = -\frac{1}{2} \cdot \frac{1}{2\pi i} \sum_{n \in \mathbb{Z}} \left(\frac{1}{x+n} + \frac{1}{x-n}\right),$$

and for $r \geq 2$,

$$h_r(t) = (r-1)! \cdot \left(-\frac{1}{2\pi i}\right)^r \sum_{n \in \mathbb{Z}} \frac{1}{(x+n)^r}.$$

Some problems for the midterm exam

m1. Compute $\left(\frac{-6}{8627}\right)$ where $\left(\frac{\cdot}{\cdot}\right)$ is the Legendre symbol.

m2. Compute $(250, -7)_5$ where $(\cdot, \cdot)_5$ is the Hilbert symbol.

m3. Compute the 7-adic expansion of 1/8.

m4. Find all pairs (x, y) with $x, y \in \mathbb{Q}$ such that $x^2 + 2y^2 = 1$.

m5. Prove that there does not exist a square root of 5 in $\mathbb{Q}_7$.

m6.1. Show that there exist two square roots of 6 in $\mathbb{Q}_5$.
m6.2. Compute the first 3 digits of their 5-adic expansions.

m7. Compute the 5-adic expansion of 1/3.

m8. Give a sequence of integers that converges to 0 in $\mathbb{Q}_3$ and to 1 in $\mathbb{Q}_5$.

m9. Consider the equation $-5x^2 - y^2 = 1$.
m9.1. Show that the equation has no solution in $\mathbb{Q}$.
m9.2. Show that the equation has a solution in $\mathbb{Q}_5$.
m9.3. Is there a prime number p other than 5 such that the equation has a solution in $\mathbb{Q}_p$? If yes, give an example.

m10. Let $a_0 + 5a_1 + 5^2 a_2 + \cdots = \exp(5)$ denote the 5-adic expansion. Compute a_0, a_1, a_2.

m11. Let $a_0 + 5a_1 + 5^2 a_2 + \cdots = \log(6)$ denote the 5-adic expansion. Compute a_0, a_1, a_2.

m12. Find all primitive roots modulo 7.

m13.1. Show that $\mathbb{Z}$ is dense in $\mathbb{Z}_p$.
m13.2. Show that $\mathbb{Q}$ is dense in $\mathbb{Q}_p$.
m13.3. Show that $\mathbb{Z}_p$ is open and closed in $\mathbb{Q}_p$.

From older problem sets

m14. Show that (the validity of) the Goldbach conjecture implies the ternary Goldbach conjecture.

m15.1. Suppose p and $q := 2p + 1$ are primes and $p \equiv 1 \pmod 4$. Show that 2 is a primitive root modulo q.
m15.2. Suppose p and $q := 4p + 1$ are odd primes. Show that 2 is a primitive root modulo q.

m16. Show $(-1, 2)_v = 1$ for all v.